6.6 The Model Existence Game

In this section we learn a new game associated with trying to construct a model for a sentence or a set of sentences. This is of fundamental importance in the sequel.

Let us first recall the game SG($\mathcal{M}, T$): The winning condition for **II** in the game SG($\mathcal{M}, T$) is the only place where the model $\mathcal{M}$ (rather than the set M) appears. If we do not start with a model $\mathcal{M}$ we can replace the winning condition with a slightly weaker one and get a very useful criterion for the existence of *some* $\mathcal{M}$ such that $\mathcal{M} \models T$:

Definition 6.29 The *Model Existence Game* $\mathrm{MEG}(T, L)$ of the set T of L-sentences in NNF is defined as follows. Let C be a countably infinite set of new constant symbols. $\mathrm{MEG}(T, L)$ is the game $G_\omega(W)$ (see Figure 6.11), where W consists of sequences $(x_0, y_0, x_1, y_1, \ldots)$ where player **II** has followed the rules of Figure 6.18 and for no atomic $L \cup C$-sentence φ both φ and $\neg\varphi$ are in $\{y_0, y_1, \ldots\}$.

The idea of the game $\mathrm{MEG}(T, L)$ is that player **I** does not doubt the truth of T (as there is no model around) but rather the mere consistency of T. So he picks those $\varphi \in T$ that he thinks constitute a contradiction and offers them to player **II** for confirmation. Then he runs through the subformulas of these sentences as if there was a model around in which they cannot all be true. He wins if he has made player **II** play contradictory basic sentences. It turns out it did not matter that we had no model around, as two contradictory sentences cannot hold in any model anyway.

Definition 6.30 Let L be a vocabulary with at least one constant symbol. A *Hintikka set (for first-order logic)* is a set H of L-sentences in NNF such that:

1. $\approx tt \in H$ for every constant L-term t.
2. If $\varphi(x)$ is basic, $\varphi(c) \in H$ and $\approx tc \in H$, then $\varphi(t) \in H$.
3. If $\varphi \wedge \psi \in H$, then $\varphi \in H$ and $\psi \in H$.
4. If $\varphi \vee \psi \in H$, then $\varphi \in H$ or $\psi \in H$.
5. If $\forall x\varphi(x) \in H$, then $\varphi(c) \in H$ for all $c \in L$
6. If $\exists x\varphi(x) \in H$, then $\varphi(c) \in H$ for some $c \in L$.
7. For every constant L-term t there is $c \in L$ such that $\approx ct \in H$.
8. There is no atomic sentence φ such that $\varphi \in H$ and $\neg\varphi \in H$.

Lemma 6.31 *Suppose L is a vocabulary and T is a set of L-sentences. If T has a model, then T can be extended to a Hintikka set.*

x_n	y_n	Explanation
φ		**I** enquires about $\varphi \in T$.
	φ	**II** confirms.
$\approx tt$		**I** enquires about an equation.
	$\approx tt$	**II** confirms.
$\varphi(t')$		**I** chooses played $\varphi(t)$ and $\approx tt'$ with φ basic and enquires about substituting t' for t in φ.
	$\varphi(t')$	**II** confirms.
φ_i		**I** tests a played $\varphi_0 \wedge \varphi_1$ by choosing $i \in \{0, 1\}$.
	φ_i	**II** confirms.
$\varphi_0 \vee \varphi_1$		**I** enquires about a played disjunction.
	φ_i	**II** makes a choice of $i \in \{0, 1\}$
$\varphi(c)$		**I** tests a played $\forall x\varphi(x)$ by choosing $c \in C$.
	$\varphi(c)$	**II** confirms.
$\exists x\varphi(x)$		**I** enquires about a played existential statement.
	$\varphi(c)$	**II** makes a choice of $c \in C$
t		**I** enquires about a constant $L \cup C$-term t.
	$\approx ct$	**II** makes a choice of $c \in C$

Figure 6.18 The game $\mathrm{MEG}(T, L)$.

Proof Let us assume $\mathcal{M} \models T$. Let $L' \supseteq L$ such that L' has a constant symbol $c_a \notin L$ for each $a \in M$. Let $\mathcal{M}^*$ be an expansion of $\mathcal{M}$ obtained by interpreting c_a by a for each $a \in M$. Let H be the set of all L'-sentences true in $\mathcal{M}$. It is easy to verify that H is a Hintikka set.

□

Lemma 6.32 *Suppose L is a countable vocabulary and T is a set of L-sentences. If player* **II** *has a winning strategy in* $\mathrm{MEG}(T, L)$*, then the set T can be extended to a Hintikka set in a countable vocabulary extending L by constant symbols.*

Proof Suppose player **II** has a winning strategy in $\mathrm{MEG}(T, L)$. We first run through one carefully planned play of $\mathrm{MEG}(T, L)$. This will give rise to a model $\mathcal{M}$. Then we play again, this time providing a proof that $\mathcal{M} \models T$. To this end, let Trm be the set of all constant $L \cup C$-terms. Let

$$\begin{aligned} T &= \{\varphi_n : n \in \mathbb{N}\}, \\ C &= \{c_n : n \in \mathbb{N}\}, \\ Trm &= \{t_n : n \in \mathbb{N}\}. \end{aligned}$$

Let $(x_0, y_0, x_1, y_1, \ldots)$ be a play in which player **II** has used her winning strategy and player **I** has maintained the following conditions:

1. If $n = 3^i$, then $x_n = \varphi_i$.
2. If $n = 2 \cdot 3^i$, then x_n is $\approx c_i c_i$.
3. If $n = 4 \cdot 3^i \cdot 5^j \cdot 7^k \cdot 11^l$, y_i is $\approx t_j t_k$, and y_l is $\varphi(t_j)$, then x_n is $\varphi(t_k)$.
4. If $n = 8 \cdot 3^i \cdot 5^j$, y_i is $\theta_0 \wedge \theta_1$, and $j < 2$, then x_n is θ_j.
5. If $n = 16 \cdot 3^i$, and y_i is $\theta_0 \vee \theta_1$, then x_n is $\theta_0 \vee \theta_1$.
6. If $n = 32 \cdot 3^i \cdot 5^j$, y_i is $\forall x \varphi(x)$, then x_n is $\varphi(c_j)$.
7. If $n = 64 \cdot 3^i$, and y_i is $\exists x \varphi(x)$, then x_n is $\exists x \varphi(x)$.
8. If $n = 128 \cdot 3^i$, then x_n is t_i.

The idea of these conditions is that player **I** challenges player **II** in a maximal way. To guarantee this he makes a plan. The plan is, for example, that on round 3^i he always plays φ_i from the set T. Thus in an infinite game every element of T will be played. Also the plan involves the rule that if player **II** happens to play a conjunction $\theta_0 \wedge \theta_1$ on round i, then player **I** will necessarily play θ_0 on round $8 \cdot 3^i$ and θ_1 on round $8 \cdot 3^i \cdot 5$, etc. It is all just book-keeping – making sure that all possibilities will be scanned. This strategy of **I** is called the *enumeration strategy*. It is now routine to show that $H = \{y_0, y_1, \ldots\}$ is a Hintikka set. □

Lemma 6.33 *Every Hintikka set has a model in which every element is the interpretation of a constant symbol.*

Proof Let $c \sim c'$ if $\approx c'c \in H$. The relation $\sim$ is an equivalence relation on C (see Exercise 6.77). Let us define an $L \cup C$-structure $\mathcal{M}$ as follows.

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We let $M = \{[c] : c \in C\}$. For $c \in C$ we let $c^{\mathcal{M}} = [c]$. If $f \in L$ and $\#(f) = n$ we let $f^{\mathcal{M}}([c_{i_1}], \ldots, [c_{i_n}]) = [c]$ for some (any – see Exercise 6.78) $c \in C$ such that $\approx c f c_{i_1} \ldots c_{i_n} \in H$. For any constant term t there is a $c \in C$ such that $\approx ct \in H$. It is easy to see that $t^{\mathcal{M}} = [c]$. For the atomic sentence $\varphi = Rt_1 \ldots t_n$ we let $\mathcal{M} \models \varphi$ if and only if φ is in H. An easy induction on φ shows that if $\varphi(x_1, \ldots, x_n)$ is an L-formula and $\varphi(d_1, \ldots, d_n) \in H$ for some $d_1 \ldots, d_n$, then $\mathcal{M} \models \varphi(d_1, \ldots, d_n)$ (see Exercise 6.79). In particular, $\mathcal{M} \models T$. □

Lemma 6.34 *Suppose L is a countable vocabulary and T is a set of L-sentences. If T can be extended to a Hintikka set in a countable vocabulary extending L, then player* **II** *has a winning strategy in* $\mathrm{MEG}(T, L)$

Proof Suppose L^* is a countable vocabulary extending L such that some Hintikka set H in the vocabulary L^* extends T. Let $C = \{c_n : n \in \mathbb{N}\}$ be a new countable set of constant symbols to be used in $\mathrm{MEG}(T, L)$. Suppose $D = \{t_n : n \in \mathbb{N}\}$ is the set of constant terms of the vocabulary L^*. The winning strategy of player **II** in $\mathrm{MEG}(T, L)$ is to maintain the condition that if y_i is $\varphi(c_1, \ldots, c_n)$, then $\varphi(t_1, \ldots, t_n) \in H$. □

We can now prove the basic element of the Strategic Balance of Logic, namely the following equivalence between the Semantic Game and the Model Existence Game:

Theorem 6.35 (Model Existence Theorem) *Suppose L is a countable vocabulary and T is a set of L-sentences. The following are equivalent:*

1. *There is an L-structure $\mathcal{M}$ such that $\mathcal{M} \models T$.*
2. *Player* **II** *has a winning strategy in* $\mathrm{MEG}(T, L)$.

Proof If there is an L-structure $\mathcal{M}$ such that $\mathcal{M} \models T$, then by Lemma 6.31 there is a Hintikka set $H \supseteq T$. Then by Lemma 6.34 player **II** has a winning strategy in $\mathrm{MEG}(T, L)$. Suppose conversely that player **II** has a winning strategy in $\mathrm{MEG}(T, L)$. By Lemma 6.32 there is a Hintikka set $H \supseteq T$. Finally, this implies by Lemma 6.33 that T has a model. □

Corollary *Suppose L is a countable vocabulary, T a set of L-sentences and φ an L-sentence. Then the following conditions are equivalent:*

1. *$T \models \varphi$.*
2. *Player* **I** *has a winning strategy in* $\mathrm{MEG}(T \cup \{\neg\varphi\}, L)$.

Proof By Theorem 3.12 the game $\mathrm{MEG}(T \cup \{\neg\varphi\}, L)$ is determined. So by Theorem 6.35, condition 2 is equivalent to $T \cup \{\neg\varphi\}$ not having a model, which is exactly what condition 1 says. □

Condition 1 of the above Corollary is equivalent to φ having a *formal proof* from T. (See Enderton (2001), or any standard textbook in logic for a definition of formal proof.) We can think of a winning strategy of player **I** in $\mathrm{MEG}(T \cup \{\neg\varphi\}, L)$ as a *semantic proof*. In the literature this concept occurs under the names *semantic tree* or *Beth tableaux*.

6.7 Applications

The Model Existence Theorem is extremely useful in logic. Our first application – The Compactness Theorem – is a kind of model existence theorem itself and very useful throughout model theory.

Theorem 6.36 (Compactness Theorem) *Suppose L is a countable vocabulary and T is a set of L-sentences such that every finite subset of T has a model. Then T has a model.*

Proof Let C be a countably infinite set of new constant symbols as needed in $\mathrm{MEG}(T, L)$. The winning strategy of player **II** in $\mathrm{MEG}(T, L)$ is the following. Suppose

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

has been played up to now, and then player **I** plays x_n. Player **II** has made sure that $T \cup \{y_0, \ldots, y_{n-1}\}$ is *finitely consistent*, i.e. each of its finite subsets has a model. Now she makes such a move y_n that $T \cup \{y_0, \ldots, y_n\}$ is still finitely consistent. Suppose this is the case and player **I** asks a confirmation for φ, where $\varphi \in T$. Now $T \cup \{y_0, \ldots, y_{n-1}, \varphi\}$ is finitely consistent as it is the same set as $T \cup \{y_0, \ldots, y_{n-1}\}$. Suppose then player **I** asks a confirmation for θ_0, where $\theta_0 \wedge \theta_1 = y_i$ for some $i < n$. If $T_0 \cup \{y_0, \ldots, y_{n-1}, \theta_0\}$ has no model, where T_0 is a finite subset of T, then surely $T_0 \cup \{y_0, \ldots, y_{n-1}\}$ has no models either, a contradiction. Suppose then player **I** asks for a decision about $\theta_0 \vee \theta_1$, where $\theta_0 \vee \theta_1 = y_i$ for some $i < n$. If $T_0 \cup \{y_0, \ldots, y_{n-1}, \theta_0\}$ has no models, where T_0 is a finite subset of T, and also $T_1 \cup \{y_0, \ldots, y_{n-1}, \theta_1\}$ has no models, where T_1 is another finite subset of T, then $T_0 \cup T_1 \cup \{y_0, \ldots, y_{n-1}\}$ has no models, a contradiction. Suppose then player **I** asks for a confirmation for $\varphi(c)$, where $\forall x\varphi(x) = y_i$ for some $i < n$ and $c \in C$. If $T_0 \cup \{y_0, \ldots, y_{n-1}, \varphi(c)\}$ has no models, where T_0 is a finite subset of T, then $T_0 \cup \{y_0, \ldots, y_{n-1}\}$ has no models either, a contradiction. Suppose then player **I** asks a decision about $\exists x\varphi(x)$, where $\exists x\varphi(x) = y_i$ for some $i < n$. Let $c \in C$ so that c does not occur in $\{y_0, \ldots, y_{n-1}\}$. We claim that $T \cup \{y_0, \ldots, y_{n-1}, \varphi(c)\}$ is finitely

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consistent. Suppose the contrary. Then there is a finite conjunction ψ of sentences in T such that

$$\{y_0, \ldots, y_{n-1}, \psi\} \models \neg\varphi(c).$$

Hence

$$\{y_0, \ldots, y_{n-1}, \psi\} \models \forall x\neg\varphi(x).$$

But this contradicts the fact that $\{y_0, \ldots, y_{n-1}, \psi\}$ has a model in which $\exists x\varphi(x)$ is true. Finally, if t is a constant term, it follows as above that there is a constant $c \in C$ such that $T \cup \{y_0, \ldots, y_{n-1}, \approx ct\}$ is finitely consistent. □

It is a consequence of the Compactness Theorem that a theory in a countable vocabulary is consistent in the sense that every finite subset has a model if and only if it is consistent in the sense that T itself has a model. Therefore the word "consistent" is used in both meanings.

As an application of the Compactness Theorem consider the vocabulary $L = \{+, \cdot, 0, 1\}$ of number theory. An example of an L-structure is the so-called *standard model of number theory* $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1)$. L-structures may be elementary equivalent to $\mathcal{N}$ and still be *non-standard* in the sense that they are not isomorphic to $\mathcal{N}$. Let c be a new constant symbol. It is easy to see that the theory

$$\{\varphi : \mathcal{N} \models \varphi\} \cup \{1 < c, +11 < c, ++111 < c, \ldots\}$$

is finitely consistent. By the Compactness Theorem it has a model $\mathcal{M}$. Clearly $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M} \not\cong \mathcal{N}$.

Example 6.37 Suppose T is a theory in a countable vocabulary L, and T has for each $n > 0$ a model $\mathcal{M}_n$ such that $(M_n, E^{\mathcal{M}_n})$ is a graph with a cycle of length $\geq n$. We show that T has a model $\mathcal{N}$ such that $(N, E^{\mathcal{N}})$ is a graph with an infinite cycle (i.e. an infinite connected subgraph in which every node has degree 2). To this end, let c_z, $z \in \mathbb{Z}$, be new constant symbols. Let T' be the theory

$$T \cup \{c_z E c_{z+1} : z \in \mathbb{Z}\}.$$

Any finite subset of T' mentions only finitely constants c_z, so it can be satisfied in the model $\mathcal{M}_n$ for a sufficiently large n. By the Compactness Theorem T' has a model $\mathcal{M}$. Now $\mathcal{M} \restriction L \models T$ and the elements $c_z^{\mathcal{M}}$, $z \in \mathbb{Z}$, constitute an infinite cycle in $\mathcal{M}$.

As another application of the Model Existence Game we prove the so-called

Thus

$$(\mathbb{N}, +, \cdot, 0, 1, A) \equiv \mathcal{M}' \equiv \mathcal{M}.$$

□

In general, the significance of the Omitting Types Theorem is the fact that it can be used – as above – to get "standard" models.

6.8 Interpolation

The Craig Interpolation Theorem says the following: Suppose $\models \varphi \to \psi$, where φ is an L_1-sentence and ψ is an L_2-sentence. Then there is an $L_1 \cap L_2$-sentence θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$. Here is an example:

Example 6.39 $L_1 = \{P, Q, R\}$, $L_2 = \{P, Q, S\}$. Let

$$\varphi = \forall x(Px \to Rx) \wedge \forall x(Rx \to Qx)$$

and

$$\psi = \forall x(Sx \to Px) \to \forall x(Sx \to Qx).$$

Now

$$\models \varphi \to \psi,$$

and indeed, if

$$\theta = \forall x(Px \to Qx),$$

then θ is an $L_1 \cap L_2$-sentence such that

$$\models \varphi \to \theta \text{ and } \models \theta \to \psi.$$

The Craig Interpolation Theorem is a consequence of the following remarkable *subformula property* of the Model Existence Game $\mathrm{MEG}(T, L)$: Player **II** never has to play anything but subformulas of sentences of T up to a substitution of terms for free variables.

Theorem 6.40 (Craig Interpolation Theorem) *Suppose $\models \varphi \to \psi$, where φ is an L_1-sentence and ψ is an L_2-sentence. Then there is an $L_1 \cap L_2$-sentence θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$.*

Proof We assume, for simplicity, that L_1 and L_2 are relational. This restriction can be avoided (see Exercise 6.97). Let us assume that the claim of the theorem is false and derive a contradiction. Since $\models \varphi \to \psi$, player **I** has a winning strategy in $\mathrm{MEG}(\{\varphi, \neg\psi\}, L_1 \cup L_2)$. Therefore to reach

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a contradiction it suffices to construct a winning strategy for player **II** in $\mathrm{MEG}(\{\varphi, \neg\psi\}, L_1 \cup L_2)$. If φ alone is inconsistent, we can take any inconsistent L-sentence as θ. Likewise if $\neg\psi$ alone is inconsistent, we can take any valid L-sentence as θ. Let $L = L_1 \cap L_2$. Let us consider the following strategy of player **II**. Suppose $C = \{c_n : n \in \mathbb{N}\}$ is a set of new constant symbols. We denote $L \cup C$-sentences by $\theta(c_0, \ldots, c_{m-1})$ where $\theta(z_0, \ldots, z_{m-1})$ is assumed to be an L-formula. Suppose player **II** has played $Y = \{y_0, \ldots, y_{n-1}\}$ so far. While she plays, she maintains two subsets S_1^n and S_2^n of Y such that $S_1^n \cup S_2^n = Y$. The set S_1^n consists of all $L_1 \cup C$-sentences in Y, and S_2^n consists of all $L_2 \cup C$-sentences in Y. Let us say that an $L \cup C$-sentence θ *separates* S_1^n and S_2^n if $S_1^n \models \theta$ and $S_2^n \models \neg\theta$. Player **II** plays so that the following condition holds at all times:

($\star$) There is no $L \cup C$-sentence θ that separates S_1^n and S_2^n.

Let us check that she can maintain this strategy: (There is no harm in assuming that player **I** plays φ and $\neg\psi$ first.)

Case 1. Player **I** plays φ. We let $S_1^0 = \{\varphi\}$ and $S_2^0 = \emptyset$. Condition ($\star$) holds, as S_1^n is consistent.

Case 2. Player **I** plays $\neg\psi$ having already played φ. We let $S_1^1 = \{\varphi\}$ and $S_2^1 = \{\neg\psi\}$. Suppose $\theta(c_0, \ldots, c_{m-1})$ separates S_1^1 and S_2^1. Then $\models \varphi \to \forall z_0 \ldots \forall z_{m-1}\theta(z_0, \ldots, z_{m-1})$ and $\models \forall z_0 \ldots \forall z_{m-1}\theta(z_0, \ldots, z_{m-1}) \to \psi$ contrary to assumption.

Case 3. Player **I** plays $\approx cc$, where, for example, $c \in L_1 \cup C$. We let $S_1^{n+1} = S_0^n \cup \{\approx cc\}$ and $S_2^{n+1} = S_1^n \cup \{\approx cc\}$. Suppose $\theta(c_0, \ldots, c_{m-1})$ separates S_1^{n+1} and S_2^{n+1}. Then clearly also $\theta(c_0, \ldots, c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 4. Player **I** plays $\varphi_0(c_1)$, where, for example, $\varphi_0(c_0), \approx c_0c_1 \in S_1^n$. We let $S_1^{n+1} = S_1^n \cup \{\varphi_0(c_1)\}$ and $S_2^{n+1} = S_2^n$. Suppose $\theta(c_0, \ldots, c_m)$ separates S_1^{n+1} and S_2^{n+1}. Then as $S_1^n \models \varphi_0(c_1)$ clearly $\theta(c_0, \ldots, c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 5. Player **I** plays φ_i, where, for example, $\varphi_1 \wedge \varphi_1 \in S_1^n$. We let $S_1^{n+1} = S_1^n \cup \{\varphi_i\}$ and $S_2^{n+1} = S_2^n$. Suppose $\theta(c_0, \ldots, c_{m-1})$ separates S_1^{n+1} and S_2^{n+1}. Then, as $S_1^n \models \varphi_i$, clearly $\theta(c_0, \ldots, c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 6. Player **I** plays $\varphi_0 \vee \varphi_1$, where, for example, $\varphi_0 \vee \varphi_1 \in S_1^n$. We claim that for one of $i \in \{0, 1\}$ the sets $S_1^n \cup \{\varphi_i\}$ and S_2^n satisfy ($\star$). Otherwise there is for both $i \in \{0, 1\}$ some $\theta_i(c_0, \ldots, c_{m-1})$ that separates $S_1^n \cup \{\varphi_i\}$

and S_2^n. Let

$$\theta(c_0,\ldots,c_{m-1}) = \theta_0(c_0,\ldots,c_{m-1}) \vee \theta_1(c_0,\ldots,c_{m-1}).$$

Then, as $S_1^n \models \varphi_0 \vee \varphi_1$, clearly $\theta(c_0,\ldots,c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 7. Player **I** plays $\varphi(c_0)$, where, for example, $\forall x\varphi(x) \in S_1^n$. We claim that the sets $S_1^n \cup \{\varphi(c_0)\}$ and S_2^n satisfy ($\star$). Otherwise there is $\theta(c_0,\ldots,c_{m-1})$ that separates $S_1^n \cup \{\varphi(c_0)\}$ and S_2^n. Let

$$\theta'(c_1,\ldots,c_{m-1}) = \forall x\theta(x,c_1,\ldots,c_{m-1}).$$

Then, as $S_1^n \models \forall x\varphi(x)$, we have $S_1^n \models \varphi(c_0)$, and hence $\theta'(c_0,c_1,\ldots,c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 8. Player **I** plays $\exists x\varphi(x)$, where, for example, $\exists x\varphi(x) \in S_1^n$. Let $c \in C$ be such that c does not occur in Y yet. We claim that the sets $S_1^n \cup \{\varphi(c)\}$ and S_2^n satisfy ($\star$). Otherwise there is some $\theta(c,c_0,\ldots,c_{m-1})$ that separates $S_1^n \cup \{\varphi(c)\}$ and S_2^n. Let

$$\theta'(c_1,\ldots,c_{m-1}) = \exists x\theta(x,c_0,\ldots,c_{m-1}).$$

Then, as $S_1^n \models \exists x\varphi(x)$ and $S_1^n \models \varphi(c) \to \theta(c,c_0,\ldots,c_{m-1})$ we clearly have that $\theta'(c_1,\ldots,c_{m-1})$ separates S_1^n and S_2^n, a contradiction. $\square$

Example 6.41 The Craig Interpolation Theorem is false in finite models. To see this, let $L_1 = \{R\}$ and $L_2 = \{P\}$ where R and P are distinct binary predicates. Let φ say that R is an equivalence relation with all classes of size 2 and let ψ say P is not an equivalence relation with all classes of size 2 except one of size 1. Then $\mathcal{M} \models \varphi \to \psi$ holds for finite $\mathcal{M}$. If there were a sentence θ of the empty vocabulary such that $\mathcal{M} \models \varphi \to \theta$ and $\mathcal{M} \models \theta \to \psi$ for all finite $\mathcal{M}$, then θ would characterize even cardinality in finite models. It is easy to see with Ehrenfeucht–Fraïssé Games that this is impossible.

Theorem 6.42 (Beth Definability Theorem) *Suppose L is a vocabulary and P is a predicate symbol not in L. Let φ be an $L \cup \{P\}$-sentence. Then the following are equivalent:*

1. *If $(\mathcal{M}, A) \models \varphi$ and $(\mathcal{M}, B) \models \varphi$, where $\mathcal{M}$ is an L-structure, then $A = B$.*
2. *There is an L-formula θ such that*

$$\varphi \models \forall x_0 \ldots x_{n-1}(\theta(x_0,\ldots,x_{n-1}) \leftrightarrow P(x_0,\ldots,x_{n-1})).$$

If condition 1 holds we say that φ defines P *implicitly*. If condition 2 holds, we say that θ defines P *explicitly* relative to φ.

Proof Let φ' be obtained from φ by replacing everywhere P by P' (another new predicate symbol). Then condition 1 implies

$$\models (\varphi \wedge Pc_0 \ldots c_{n-1}) \to (\varphi' \to P'c_0 \ldots c_{n-1}).$$

By the Craig Interpolation Theorem there is an L-formula $\theta(x_0,\ldots,x_{n-1})$ such that

$$\models (\varphi \wedge Pc_0 \ldots c_{n-1}) \to \theta(c_0,\ldots,c_{n-1})$$

and

$$\models \theta(c_0,\ldots,c_{n-1}) \to (\varphi' \to P'c_0 \ldots c_{n-1}).$$

It follows easily that θ is the formula we are looking for. $\square$

Example 6.43 The Beth Definability Theorem is false in finite models. Let φ be the conjunction of

1. "$<$ is a linear order".
2. $\exists x(Px \wedge \forall y(\approx xy \vee x < y))$.
3. $\forall x\forall y$("y immediate successor of x" $\to (Px \leftrightarrow \neg Py)$).

Every finite linear order has a unique P with φ, but there is no $\{<\}$-formula $\theta(x)$ which defines P in models of φ. For then the sentence

$$\exists x(\theta(x) \wedge \forall y(\approx xy \vee y < x))$$

would characterize ordered sets of odd length among finite ordered sets, and it is easy to see with Ehrenfeucht–Fraïssé Games that no such sentence can exist. There are infinite linear orders (e.g. $(\mathbb{N} + \mathbb{Z}, <)$) where several different P satisfy φ.

Recall that the *reduct* of an L-structure $\mathcal{M}$ to a smaller vocabulary K is the structure $\mathcal{N} = \mathcal{M} \restriction K$ which has M as its universe and the same interpretations of all symbols of K as $\mathcal{M}$. In such a case we call $\mathcal{M}$ an *expansion* of $\mathcal{N}$ from vocabulary K to vocabulary L. Another useful operation on structures is the following. The *relativization* of an L-structure $\mathcal{M}$ to a set N is the structure $\mathcal{N} = \mathcal{M}^{(N)}$ which has N as its universe, $R^{\mathcal{M}} \cap N^{\#(R)}$ as the interpretation of any predicate symbol $R \in L$, $f^{\mathcal{M}} \restriction N^{\#(f)}$ as the interpretation of any function symbol $f \in L$, and $c^{\mathcal{M}}$ as the interpretation of any constant symbol $c \in L$. Relativization is only possible when the result actually *is* an L-structure. There is a corresponding operation on formulas: The *relativization* of an L-formula φ to a predicate $P \in L$ is defined by replacing every quantifier $\forall y \ldots$ in φ by $\forall y(Py \to \ldots)$ and every quantifier $\exists y \ldots$ in φ by $\exists y(Py \wedge \ldots)$. We denote the relativization by $\psi^{(P)}$.

Lemma 6.44 *Suppose L is a relational vocabulary and $P \in L$ is a unary predicate symbol. The following are equivalent for all L-formulas φ and all L-structures $\mathcal{M}$ such that $P^{\mathcal{M}} \neq \emptyset$:*

1. $\mathcal{M} \models \varphi^{(P)}$.
2. $\mathcal{M}^{(P^{\mathcal{M}})} \models \varphi$.

Proof Exercise 6.101. □

Definition 6.45 Suppose L is a vocabulary. A class K of L-structures is an *EC-class* if there is an L-sentence φ such that

$$K = \{\mathcal{M} \in \mathrm{Str}(L) : \mathcal{M} \models \varphi\}$$

and a *PC-class* if there is an L'-sentence φ for some $L' \supseteq L$ such that

$$K = \{\mathcal{M} \restriction L : \mathcal{M} \in \mathrm{Str}(L') \text{ and } \mathcal{M} \models \varphi\}.$$

Example 6.46 Let $L = \emptyset$. The class of infinite L-structures is a PC-class which is not an EC class. (Exercise 6.102.)

Example 6.47 Let $L = \emptyset$. The class of finite L-structures is not a PC-class. (Exercise 6.103.)

Example 6.48 Let $L = \{<\}$. The class of non-well-ordered L-structures is a PC-class which is not an EC-class. (Exercise 6.104.)

Suppose $\models \varphi \to \psi$, where φ is an L_1-sentence and ψ is an L_2-sentence. Let

$$K_1 = \{\mathcal{M} \restriction (L_1 \cap L_2) : \mathcal{M} \models \varphi\}$$

and

$$K_2 = \{\mathcal{M} \restriction (L_1 \cap L_2) : \mathcal{M} \models \neg\psi\}.$$

Now K_1 and K_2 are disjoint PC-classes. If there is an $L_1 \cap L_2$-sentence θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$, then the EC-class

$$K = \{\mathcal{M} : \mathcal{M} \models \theta\}$$

separates K_1 and K_2 in the sense that $K_1 \subseteq K$ and $K_2 \cap K = \emptyset$. On the other hand, if an EC-class K separates in this sense K_1 and K_2, then there is an $L_1 \cap L_2$-sentence θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$. Thus the Craig Interpolation Theorem can be stated as: disjoint PC-classes can always be separated by an EC-class.

Theorem 6.49 (Separation Theorem) *Suppose K_1 and K_2 are disjoint PC-classes of models. Then there is an EC-class K that separates K_1 and K_2, i.e. $K_1 \subseteq K$ and $K_2 \cap K = \emptyset$.*

Proof The claim has already been proved in Theorem 6.40, but we give here a different – model-theoretic – proof. This proof is of independent interest, being as it is, in effect, the proof of the so-called *Lindström's Theorem* (Lindström (1973)), which gives a model theoretic characterization of first order logic.

Case 1: There is an $n \in \mathbb{N}$ such that some union K of $\simeq_p^n$-equivalence classes of models separates K_1 and K_2. By Theorem 6.5 the model class K is an EC-class, so the claim is proved.

Case 2: There are, for any $n \in \mathbb{N}$, $L_1 \cap L_2$-models $\mathcal{M}_n$ and $\mathcal{N}_n$ such that $\mathcal{M}_n \in K_1$, $\mathcal{N}_n \in K_2$, and there is a back-and-forth sequence $(I_i : i \leq n)$ for $\mathcal{M}_n$ and $\mathcal{N}_n$. Suppose K_1 is the class of reducts of models of φ, and K_2 respectively the class of reducts of models of ψ. Let T be the following set of sentences:

1. $\varphi^{(P_1)}$.
2. $\psi^{(P_2)}$.
3. $(R, <)$ is a non-empty linear order in which every element with a predecessor has an immediate predecessor.
4. $\forall z(Rz \to Q_0 z)$.
5. $\forall z \forall u_1 \ldots \forall u_m \forall v_1 \ldots \forall v_m((Rz \wedge Q_n z u_1 \ldots u_m v_1 \ldots v_m) \to (\theta(u_1, \ldots, u_m) \leftrightarrow \theta(v_1, \ldots, v_m)))$ for all atomic $L_1 \cap L_2$-formulas θ.
6. $\forall z \forall u_1 \ldots \forall u_n \forall v_1 \ldots \forall v_m((Rz \wedge Q_n z u_1 \ldots u_m v_1 \ldots v_m) \to \forall z' \forall x((Rz' \wedge z' < z \wedge \forall w(w < z \to (w < z' \vee w = z')) \wedge P_1 x) \to \exists y(P_2 y \wedge Q_{n+1} z' u_1 \ldots u_m x v_1 \ldots v_m y)))$.
7. $\forall z \forall u_1 \ldots \forall u_m \forall v_1 \ldots \forall v_m((Rz \wedge Q_n z u_1 \ldots u_m v_1 \ldots v_m) \to \forall z' \forall y((Rz' \wedge z' < z \wedge \forall w(w < z \to (w < z' \vee w = z')) \wedge P_2 y) \to \exists x(P_1 x \wedge Q_{n+1} z' u_1 \ldots u_m x v_1 \ldots v_m y)))$.

For all $n \in \mathbb{N}$ there is a model $\mathcal{A}_n$ of T with $(R, <)$ of length n. The model $\mathcal{A}_n$ is obtained as follows. The universe A_n is the (disjoint) union of M_n, N_n, and $\{1, \ldots, n\}$. The L_1-structure $(\mathcal{A}_n \restriction L_1)^{P_1^{\mathcal{A}_n}}$ is chosen to be a copy of the model $\mathcal{M}_n$ of φ. The L_2-structure $(\mathcal{A}_n \restriction L_2)^{P_2^{\mathcal{A}_n}}$ is chosen to be a copy of the model $\mathcal{N}_n$ of ψ. The $2i+1$-ary predicate Q_i is interpreted in $\mathcal{A}_n$ as the set

$$\{(n-i, u_1, \ldots, u_i, v_1, \ldots, v_i) : \{(u_1, v_1), \ldots, (u_i, v_i)\} \in I_{n-i}\}.$$

By the Compactness Theorem, there is a countable model $\mathcal{M}$ of T with $(R, <)$ non-well-founded (see Exercise 6.107). That is, there are a_n, $n \in \mathbb{N}$, in M such that a_{n+1} is an immediate predecessor of a_n in $\mathcal{M}$ for all $n \in \mathbb{N}$. Let $\mathcal{M}_1$ be the $L_1 \cap L_2$-structure $(\mathcal{M} \restriction (L_1 \cap L_2))^{(P_1^{\mathcal{M}})}$. Let $\mathcal{M}_2$ be the $L_1 \cap L_2$-structure $(\mathcal{M} \restriction (L_1 \cap L_2))^{(P_2^{\mathcal{M}})}$. Now $\mathcal{M}_1 \simeq_p \mathcal{M}_2$, for we have the back-and-forth set:

$$P = \{\{(u_1, v_1), \ldots, (u_n, v_n)\} : \mathcal{M} \models Q_n a_n u_1 \ldots u_n v_1 \ldots v_n, n \in \mathbb{N}\}.$$