Cambridge University Press 2007

Jouko Väänänen

Dependence Logic

A New Approach to Independence Friendly Logic

Chapter 1

Dependence Logic

Dependence logic introduces the concept of *dependence* into first order logic by adding a new kind of atomic formula. We call these new atomic formulas *atomic dependence formulas*. The definition of the semantics for dependence logic is reminiscent of the definition of the semantics for first order logic. But instead of defining satisfaction for assignments, we follow [12] and jump one level up considering *sets* of assignments. This leads us to formulate the semantics of dependence logic in terms of the concept of the *type* of a set of assignments.

The reason for the transition to a higher level is, roughly speaking, that one cannot manifest dependence, or independence for that matter, in a single assignment. To see a pattern of dependence one needs a whole set of assignments.

This is because dependence notions can be best investigated in a context involving repeated actions by agents presumably governed by some possibly hidden rules. In such a context dependence is manifested by recurrence, and independence by lack of it.

Our framework consists of three components: teams, agents, and features. Teams are sets of agents. Agents are objects with features. Features are like variables which can have any value in a given fixed set.

Although our treatment of dependence logic is entirely mathematical, our intuition of dependence phenomena comes from real life examples, thinking of different ways dependence manifests itself in the real world. Statisticians certainly have much to say about this but when we go deeper into the logic of dependence we see that the crucial concept is determination, not mere dependence. Another difference with statistics is that we study total dependence, not statistically significant dependence. It would seem reasonable to define probabilistic dependence logic, but we will not go into that here.

1.1 Examples and a Mathematical Model for Teams

In practical examples a feature is anything that can be in the domain of a function: color, length, weight, prize, profession, salary, gender, etc. To be specific, we use variables $x_0, x_1, ...$ to denote features. If features are variables then agents are *assignments*. When we define dependence logic, we use the variable x_n to refer to the value $s(x_n)$ of the feature x_n in an agent s.

1. Consider teams of soccer players. In this case the players are the agents. The number of the player as well as the colors of their shirts and pants are the features, denoted by variables x_0, x_1, x_2, respectively. Teams are sets of players in the usual sense of the word "team." Figure 1.1 depicts a team. If we counted only the color of

	(player) x_0	(shirt) x_1	(pants) x_2
s_0	1	yellow	white
s_1	2	yellow	white
s_2	3	yellow	white
s_3	4	yellow	white
s_4	5	red	white
s_5	6	red	black
s_6	7	red	black

Figure 1.1: Soccer players as a team.

   the players' shirts and pants as features, we would get the generated team of three agents depicted in Figure 1.2.

2. Databases are good examples of teams. By a database we mean in this context a table of data arranged in columns and rows. The

	(shirt) x_1	(pants) x_2
s_0	yellow	white
s_1	red	white
s_2	red	black

Figure 1.2: A generated team.

columns are the features, the rows are the agents, and the set consisting of the rows is the team. In database theory the columns are often called fields or attributes, and the rows are called records or tuples. Figure 1.3 is an example of a database. If the row number (1 to k in Figure 1.3) is counted as a feature, then all rows are different agents. Otherwise rows with identical values in all the features are identified, resulting an a team called the *generated team*, i.e. the team generated by the particular database. Figure 1.4 depicts a database, arising from a game, and the generated team.

	Fields			
Record	x_1	x_2	...	x_n
1	52	24	...	1
2	68	362	...	0
3	11	7311	...	1
...	...	...	...	...
k	101	43	...	1

Figure 1.3: A database.

3. The game history team: Imagine a game where players make moves, following a strategy they have chosen with a certain goal in mind. We are thinking of games in the sense of von Neumann and Morgenstern "Theory of Games and Economic Behavior" [29]. Examples of such games are board and card games, business games, games related to social behavior, etc. We think of the moves of the game as features. If during a game 350 moves are made by the players, then we have 350 features. Plays, i.e. sequences of moves of the game that comprise an entire play of the game, are the agents. Any collection of plays is

a team. A team may arise for example as follows: Two players play a certain game 25 times thus producing 25 sequences of moves. A team of 25 agents is created.

It may be desirable to know answers to the following kinds of questions:

(a) What is the strategy that a player is following, or is he or she following any strategy at all?

(b) Is a player using information about his or her (or other players') moves that he or she is committed not to use? This issue is closely related to game-theoretic semantics of dependence logic treated in Chapter 3.

The following game illustrates how a player can use information that may be not admitted: There are two players **I** and **II**. Player **I** starts by choosing an integer n. Then **II** chooses an integer m. After this **II** makes another move and chooses, this time without seeing n, an integer l. So player **II** is committed to choose l without seeing n even if she saw n when she picked m. One may ask, how can she forget a number she has seen once, but if the number has many digits this is quite plausible. Player **II** wins if $l > n$. In other words, **II** has the impossible looking task of choosing an integer l which is bigger than an integer n that she is not allowed to know. Her trick, which we call the signalling-strategy, is to store information about n into m and then choose l only on the basis of what m is. Figure 1.4 shows an example of a game history team in this game. We can see that player **II** has been using the signalling-strategy. If we instead observed the behavior of Figure 1.5, we could doubt whether **II** is obeying the rules, as her second move seems clearly to depend on the move of **I** which she is not supposed to see.

4. Every formula $\phi(x_1, ..., x_n)$ of any logic and structure $\mathcal{M}$ together give rise to the team of all assignments that satisfy $\phi(x_1, ..., x_n)$ in

Play	**I**	**II**	**II**
1	1	1	2
2	40	40	41
3	2	2	3
4	0	0	1
5	1	1	2
6	2	2	3
7	40	40	41
8	100	100	101

	x_0	x_1	x_2
S_0	0	0	1
S_1	1	1	2
S_2	2	2	3
S_3	40	40	41
S_4	100	100	101

Figure 1.4: A game history and the generated team.

Play	**I**	**II**	**II**
1	1	0	2
2	40	0	41
3	2	0	3
4	0	0	1
5	1	0	2
6	2	0	3
7	40	0	41
8	100	0	101

	x_0	x_1	x_2
s_0	0	0	1
s_1	1	0	2
s_2	2	0	3
s_3	40	0	41
s_4	100	0	101

Figure 1.5: A suspicious game history and the generated team.

$\mathcal{M}$. In this case the variables are the features and the assignments are the agents. This (possibly quite large) team manifests the dependence structure $\phi(x_1, ..., x_n)$ expresses in $\mathcal{M}$. If ϕ is the first order formula $x_0 = x_1$, then ϕ expresses the very strong dependence of x_1 on x_0, namely of x_1 being equal to x_0. The team of assignments satisfying $x_0 = x_1$ in a structure is the set of assignments s which give to x_0 the same value as to x_1. If ϕ is the infinitary formula $(x_0 = x_1) \vee (x_0 \cdot x_0 = x_1) \vee (x_0 \cdot x_0 \cdot x_0 = x_1) \vee ...$, then ϕ expresses the dependence of x_1 on x_0 of being in the set $\{x_0, x_0 \cdot x_0, x_0 \cdot x_0 \cdot x_0, ...\}$. See Figure **??**.

5. Every first order sentence ϕ and structure $\mathcal{M}$ together give rise to teams consisting of assignments that arise in the semantic game (see Section 3.1) of ϕ and $\mathcal{M}$. The semantic game is a game for two players **I** and **II**, in which **I** tries to show that ϕ is not true in $\mathcal{M}$, and **II** tries to show that ϕ is indeed true in $\mathcal{M}$. The game proceeds according to the structure of ϕ. At conjunctions player **I** chooses a conjunct. At universal quantifiers player **I** chooses a value for

the universally bound variable. At disjunctions player **II** chooses a disjunct. At existential quantifiers player **II** picks up a value for the existentially bound variable. At negations the players exchange roles. Thus the players build up move by move an assignment s. When an atomic formula is met, player **II** wins if the formula is true in $\mathcal{M}$ under the assignment s, otherwise player **I** wins. See Section 3.1 for details. If $M \models \phi$ and the winning strategy of II is τ in this semantic game, a particularly interesting team consists of all plays of the semantic game in which II uses τ. This team is interesting because the strategy τ can be read off from the team. We can view the study of teams of plays in this game as a generalization of the study of who wins the semantic game. The semantic game of dependence logic is treated in Chapter 3.

We now give a mathematical model for teams:

Definition 1 *An* agent *is any function s from a finite set* $\mathrm{dom}(s)$ *of variables, also called* features*, to a fixed set M. The set* $\mathrm{dom}(s)$ *is called the* domain *of s, and the set M is called the* codomain *of s. A* team *is any set X of agents with the same domain, called the* domain of X *and denoted by* $\mathrm{dom}(X)$*, and the same codomain, likewise called the* codomain of X*. A team with codomain M is called a team* of M*. If V is a finite set of variables, we use* $\mathrm{Team}(M, V)$ *to denote the set of all teams of M with domain V.*

Since we have defined teams as *sets*, not *multisets*, of assignments, one assignment can occur only once in a team. Allowing multisets would, however, change nothing essential in this study.

1.2 Formulas as Types of Teams

We define a logic which has an atomic formula for expressing dependence. We call this logic the **dependence logic** and denote it by

$\mathcal{D}$. We will later in Section 1.6 recover independence friendly logic as a fragment of dependence logic.

In first order logic the meaning of a formula is derived from the concept of an assignment satisfying the formula. In dependence logic the meaning of a formula is based on the concept of a team being of the *(dependence) type* of the formula.

Recall that teams are sets of agents (assignments) and that agents are functions from a finite set of natural numbers, called the domain of the agent into an arbitrary set called the codomain of the agent (Definition 1). In a team the domain of all agents is assumed to be the same set of natural numbers, just as the codomain of all agents is assumed to be the same set.

Our atomic dependence formulas have the form $=(t_1, ..., t_n)$. The intuitive meaning of this is "the value of the term t_n depends only on the values of the terms t_1,...,t_{n-1}." As singular cases we have $=()$, which we take to be universally true, and $=(t)$, which declares that the value of the term t depends on nothing, i.e. is constant. Note that $=(x_1)$ is quite non-trivial and indispensable if we want to say that all agents are similar as far as feature x_1 is concerned. Such similarity is manifested by the team of Figure 1.5, where all agents have value 0 in their feature x_1.

Actually, our atomic formulas express determination rather than dependence. The reason for this is that determination is a more basic concept than dependence. Once we can express determination, we can define dependence and independence. Already dependence logic has considerable strength. Further extensions formalizing the concepts of dependence and independence are even stronger, and in addition lack many of the nice model-theoretic properties that our dependence logic enjoys. We will revisit the concepts of dependence and particularly independence later below.

Definition 2 *Suppose L is a vocabulary. If $t_1, ..., t_n$ are L-terms* *and R is a relation symbol in L with arity n, then strings $t_i = t_j$,$=(t_1, ..., t_n)$,$Rt_1...t_n$ are L-formulas of dependence logic $\mathcal{D}$. They are called* atomic *formulas. If ϕ and ψ are L-formulas of $\mathcal{D}$, then $(\phi \vee \psi)$ and $\neg\phi$ are L-formulas of $\mathcal{D}$. If ϕ is an L-formula of $\mathcal{D}$ and $n \in \mathbb{N}$, then $\exists x_n \phi$ is an L-formula of $\mathcal{D}$.*

As is apparent from Definition 2, the syntax of dependence logic $\mathcal{D}$ is very similar to that of first order logic, the only difference being the inclusion of the new atomic formulas $=(t_1, ..., t_n)$. We use $(\phi \wedge \psi)$ to denote $\neg(\neg\phi \vee \neg\psi)$, $(\phi \rightarrow \psi)$ to denote $(\neg\phi \vee \psi)$, $(\phi \leftrightarrow \psi)$ to denote $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$, and $\forall x_n \phi$ to denote $\neg\exists x_n \neg\phi$. A formula of dependence logic which does not contain any atomic formulas of the form $=(t_1, ..., t_n)$ is called *first order*. The *veritas* symbol $\top$ is definable as $=()$.

The set $\mathrm{Fr}(\phi)$ of *free variables* of a formula s is defined otherwise as for first order logic, except that we have the new case $\mathrm{Fr}(=(t_1, ..., t_n)) = \mathrm{Var}(t_1) \cup ... \cup \mathrm{Var}(t_n)$. If $\mathrm{Fr}(\phi) = \emptyset$, we call ϕ an *L-sentence* of dependence logic.

We define now two important operations on teams, the supplement and the duplication operations. The supplement operation adds a new feature to the agents in a team, or alternatively changes the value of an existing feature.

Definition 3 *If M is a set, X is a team with M as its codomain and $F : X \rightarrow M$, we let $X(F/x_n)$ denote the* supplement *team $\{s(F(s)/x_n) : s \in X\}$.*

A duplicate team is obtained by duplicating agents of a team until all possibilities occur as far as a particular feature is concerned.

Definition 4 *If M is a set, X a team of M we use $X(M/x_n)$ to denote the* duplicate *team $\{s(a/x_n) : a \in M, s \in X\}$.*

We are ready to define the semantics of dependence logic:

Definition 5 *Let the set $\mathcal{T}$ be the smallest set that satisfies:*

E1 $(t_1 = t_2, X, 1) \in \mathcal{T}$ *iff for all* $s \in X$ *we have* $t_1^{\mathcal{M}}\langle s\rangle = t_2^{\mathcal{M}}\langle s\rangle$.

E2 $(t_1 = t_2, X, 0) \in \mathcal{T}$ *iff for all* $s \in X$ *we have* $t_1^{\mathcal{M}}\langle s\rangle \neq t_2^{\mathcal{M}}\langle s\rangle$.

E3 $(=(t_1, ..., t_n), X, 1)) \in \mathcal{T}$ *iff for all* $s, s' \in X$ *such that* $t_1^{\mathcal{M}}\langle s\rangle = t_1^{\mathcal{M}}\langle s'\rangle$,...,$t_{n-1}^{\mathcal{M}}\langle s\rangle = t_{n-1}^{\mathcal{M}}\langle s'\rangle$, *we have* $t_n^{\mathcal{M}}\langle s\rangle = t_n^{\mathcal{M}}\langle s'\rangle$.

E4 $(=(t_1, ..., t_n), X, 0) \in \mathcal{T}$ *iff* $X = \emptyset$.

E5 $(Rt_1...t_n, X, 1) \in \mathcal{T}$ *iff for all* $s \in X$ *we have* $(t_1^{\mathcal{M}}\langle s\rangle, ..., t_n^{\mathcal{M}}\langle s\rangle) \in R^{\mathcal{M}}$.

E6 $(Rt_1...t_n, X, 0) \in \mathcal{T}$ *iff for all* $s \in X$ *we have* $(t_1^{\mathcal{M}}\langle s\rangle, ..., t_n^{\mathcal{M}}\langle s\rangle) \notin R^{\mathcal{M}}$.

E7 $(\phi \vee \psi, X, 1) \in \mathcal{T}$ *iff* $X = Y \cup Z$ *such that* $\mathrm{dom}(Y) = \mathrm{dom}(Z)$, $(\phi, Y, 1) \in \mathcal{T}$ *and* $(\psi, Z, 1) \in \mathcal{T}$.

E8 $(\phi \vee \psi, X, 0) \in \mathcal{T}$ *iff* $(\phi, X, 0) \in \mathcal{T}$ *and* $(\psi, X, 0) \in \mathcal{T}$.

E9 $(\neg\phi, X, 0) \in \mathcal{T}$ *iff* $(\phi, X, 1) \in \mathcal{T}$.

E10 $(\neg\phi, X, 1) \in \mathcal{T}$ *iff* $(\phi, X, 0) \in \mathcal{T}$.

E11 $(\exists x_n\phi, X, 1) \in \mathcal{T}$ *iff* $(\phi, X(F/x_n), 1) \in \mathcal{T}$ *for some* $F : X \to M$.

E12 $(\exists x_n\phi, X, 0) \in \mathcal{T}$ *iff* $(\phi, X(M/x_n), 0) \in \mathcal{T}$.

We define X is of type ϕ *in $\mathcal{M}$, denoted* $\mathcal{M} \models_X \phi$ *if* $(\phi, X, 1) \in \mathcal{T}$. *Furthermore, ϕ is* true *in $\mathcal{M}$, denoted* $\mathcal{M} \models \phi$, *if* $\mathcal{M} \models_{\{\emptyset\}} \phi$, *and ϕ is valid, denoted* $\models \phi$, *if* $\mathcal{M} \models \phi$ *for all* $\mathcal{M}$.

Note that,

$$\begin{array}{lll} \mathcal{M} \models_X \neg\phi & \text{if} & (\phi, X, 0) \in \mathcal{T} \\ \mathcal{M} \models \neg\phi & \text{if} & (\phi, \{\emptyset\}, 0) \in \mathcal{T}. \text{ Then we say that } \phi \text{ is } \textit{false} \text{ in } \mathcal{M}. \end{array}$$

We will see in a moment that it is not true in general that $(\phi, X, 1) \in \mathcal{T}$ or $(\phi, X, 0) \in \mathcal{T}$. Likewise it is not true in general that $\mathcal{M} \models \phi$ or

$\mathcal{M} \models \neg\phi$, nor that $\mathcal{M} \models \phi \vee \neg\phi$. In other words, no sentence can be both true and false in a model but some sentences can be neither true nor false in a model. This gives our logic a nonclassical flavor.

Example 6 *Let $\mathcal{M}$ be a structure with $M = \{0,1\}$. Consider the team*

	x_0	x_1	x_2	x_3
s_0	*0*	*0*	*0*	*1*
s_1	*1*	*0*	*1*	*0*
s_2	*0*	*0*	*0*	*1*

This team is of type $=(x_1)$, since $s_i(x_1) = 0$ for all i. This team is of type $x_0 = x_2$, as $s_i(x_0) = s_i(x_2)$ for all i. This team is of type $\neg x_0 = x_3$, as $s_i(x_0) \neq s_i(x_3)$ for all i. This team is of type $=(x_0, x_1)$, as $s_i(x_0) = s_j(x_0)$ implies $s_i(x_3) = s_j(x_3)$. This team is not of type $=(x_1, x_2)$, as $s_0(x_1) = s_1(x_1)$, but $s_0(x_2) \neq s_1(x_2)$. Finally, this team is of type $=(x_0) \vee =(x_0)$ as it can be represented as the union $\{s_0, s_2\} \cup \{s_1\}$, where $\{s_0, s_2\}$ and $\{s_1\}$ both are of type $=(x_0)$.

Note that

- $(\phi \wedge \psi, X, 1) \in \mathcal{T}$ iff $(\phi, X, 1) \in \mathcal{T}$ and $(\psi, X, 1) \in \mathcal{T}$.
- $(\phi \wedge \psi, X, 0) \in \mathcal{T}$ iff $X = Y \cup Z$ such that $\mathrm{dom}(Y) = \mathrm{dom}(Z)$, $(\phi, Y, 0) \in \mathcal{T}$ and $(\psi, Z, 0) \in \mathcal{T}$.
- $(\forall x_n \phi, X, 1) \in \mathcal{T}$ iff $(\phi, X(M/x_n), 1) \in \mathcal{T}$.
- $(\forall x_n \phi, X, 0) \in \mathcal{T}$ iff $(\phi, X(F/x_n), 0) \in \mathcal{T}$ for some $F : X \to M$.

It may seem strange to define (D4) as $(=(t_1, ..., t_n), \emptyset, 0) \in \mathcal{T}$. Why not allow $(=(t_1, ..., t_n), X, 0) \in \mathcal{T}$ for non-empty X? The reason is that if we negate "for all $s, s' \in X$ such that $t_1^{\mathcal{M}}\langle s\rangle = t_1^{\mathcal{M}}\langle s'\rangle$,...,$t_{n-1}^{\mathcal{M}}\langle s\rangle = t_{n-1}^{\mathcal{M}}\langle s'\rangle$, we have $t_n^{\mathcal{M}}\langle s\rangle = t_n^{\mathcal{M}}\langle s'\rangle$," maintaining analogy with (D2) and (D6), we get "for all $s, s' \in X$ we have $t_1^{\mathcal{M}}\langle s\rangle = t_1^{\mathcal{M}}\langle s'\rangle$,...,$t_{n-1}^{\mathcal{M}}\langle s\rangle = t_{n-1}^{\mathcal{M}}\langle s'\rangle$ and $t_n^{\mathcal{M}}\langle s\rangle \neq t_n^{\mathcal{M}}\langle s'\rangle$," which is only possible if $X = \emptyset$.

Some immediate observations can be made using Definition 5. We first note that the empty team $\emptyset$ is of the type of any formula, as $(\phi, \emptyset, 1) \in \mathcal{T}$ holds for all ϕ. In fact:

Lemma 7 *For all ϕ and $\mathcal{M}$ we have $(\phi, \emptyset, 1) \in \mathcal{T}$ and $(\phi, \emptyset, 0) \in \mathcal{T}$.*

Proof. Inspection of definition 5 reveals that all the necessary implications hold vacuously when $X = \emptyset$.$\Box$

In other words, the empty team is for all ϕ of type ϕ and of type $\neg\phi$. Since the type of a team is defined by reference to all agents in the team, the empty team ends up having all types, just as it is usually agreed that the intersection of an empty collection of subsets of a set M is the set M itself. A consequence of this is that there are no formulas ϕ and ψ of dependence logic such that $\mathcal{M} \models_X \phi$ implies $\mathcal{M} \not\models_X \psi$, for all $\mathcal{M}$ and all X. Namely, letting $X = \emptyset$ would yield a contradiction.

The following test is very important and will be used repeatedly in the sequel. Closure downwards is a fundamental property of types in dependence logic.

Proposition 8 (Closure Test) *Suppose $Y \subseteq X$. Then $\mathcal{M} \models_X \phi$ implies $\mathcal{M} \models_Y \phi$.*

Proof. Every condition from E1 to E12 is closed under taking a subset of X. So if $(\phi, X, 1) \in \mathcal{T}$ and $Y \subseteq X$, then $(\phi, Y, 1) \in \mathcal{T}$. $\Box$

The intuition behind the Closure Test is the following: To witness the failure of dependence we need a counterexample, two or more assignments that manifest the failure. The smaller the team the fewer counterexamples. In a one-agent team no counterexample to dependence is any more possible. On the other hand, the bigger the team, the more likely it is that some lack of dependence becomes exposed. In the maximal team of all possible assignments no dependence is possible, unless the universe has just one element.

Corollary 9 *There is no formula ϕ of dependence logic such that for all $X \neq \emptyset$ and all $\mathcal{M}$ we have $\mathcal{M} \models_X \phi \iff \mathcal{M} \not\models_X =\!(x_0, x_1)$.*

Proof. Suppose for a contradiction $\mathcal{M}$ has at least two elements a, b. Let X consist of $s = \{(x_0, a), (x_1, a)\}$ and $s' = \{(x_0, a), (x_1, b)\}$. Now $\mathcal{M} \not\models_X =\!(x_0, x_1)$, so $\mathcal{M} \models_X \phi$. By Closure Test, $\mathcal{M} \models_{\{s\}} \phi$, whence $\mathcal{M} \not\models_{\{s\}} =\!(x_0, x_1)$. But this is clearly false. $\square$

We can replace "all $\mathcal{M}$" by "some $\mathcal{M}$ with more than one element in the universe" in the above corollary. Note that in particular we do not have for all $X \neq \emptyset$: $\mathcal{M} \models_X \neg =\!(x_0, x_1) \iff \mathcal{M} \not\models_X =\!(x_0, x_1)$.

Example 10 *Every team X, the domain of which contains x_i and x_j is of type $x_i = x_j \vee \neg x_i = x_j$, as we can write $X = Y \cup Z$, where $Y = \{s \in X : s(x_i) = s(x_j)\}$ and $Z = \{s \in X : s(x_i) \neq s(x_j)\}$. Note that then Y is of type $x_i = x_j$, and Z is of type $x_i \neq x_j$.*

It is important to take note of a difference between universal quantification in first order logic and universal quantification in dependence logic. It is perfectly possible to have a formula $\phi(x_0)$ of dependence logic of the empty vocabulary with just x_0 free such that for a new constant symbol c we have $\models \phi(c)$ and still $\not\models \forall x_0 \phi(x_0)$, as the following example shows. For this example, remember that $=\!(x_1)$ is the type "x_1 is constant" of a team in which all agents have the same value for their feature x_1.

Example 11 *Suppose $\mathcal{M}$ is a model of the empty*[1] *vocabulary with at least two elements in its domain. Let ϕ be the sentence $\exists x_1(=\!(x_1) \wedge c = x_1)$ of dependence logic. Then*

$$(\mathcal{M}, a) \models \exists x_1(=\!(x_1) \wedge c = x_1) \tag{1.1}$$

for all expansions of $(\mathcal{M}, a)$ of $\mathcal{M}$ to the vocabulary $\{c\}$. To prove (1.1) suppose we are given an element $a \in M$. We can define $F_a(\emptyset) = a$ and then $(\mathcal{M}, a) \models_{\{\{(x_1,a)\}\}} (=\!(x_1) \wedge c = x_1)$, where we have used $\{\emptyset\}(F_a/x_1) = \{\{(x_1, a)\}\}$. However $\mathcal{M} \not\models \forall x_0 \exists x_1(=\!(x_1) \wedge x_0 = x_1)$. To prove this suppose the contrary, that is $\mathcal{M} \models_{\{\emptyset\}}$

[1] The empty vocabulary has no constant, relation or function symbols. Structures for the empty vocabulary consists of merely a non-empty set as the universe.

$\forall x_0 \exists x_1(=\!(x_1) \wedge x_0 = x_1)$. Then $\mathcal{M} \models_{\{\{(x_0,a)\}:a\in M\}} \exists x_1(=\!(x_1) \wedge x_0 = x_1)$, where we have written $\{\emptyset\}(M/x_0)$ out as $\{\{(x_0,a)\} : a \in M\}$. Let $F : \{\{(x_0,a)\} : a \in M\} \to M$ such that $\mathcal{M} \models_{\{\{(x_0,a),(x_1,G(a))\}:a\in M\}} (=\!(x_1) \wedge x_0 = x_1)$, where $G(a) = F(\{(x_0,a)\})$ and $\{\{(x_0,a)\} : a \in M\}(F/x_1)$ has been written as $\{\{(x_0,a),(x_1,G(a))\} : a \in M\}$. In particular $\mathcal{M} \models_{\{\{(x_0,a),(x_1,G(a))\}:a\in M\}} =\!(x_1)$, which means that F has to have a constant value. Since M has at least two elements, the fact $\mathcal{M} \models_{\{\{(x_0,a),(x_1,G(a))\}:a\in M\}} x_0 = x_1$ contradicts (D1).

Exercise 1 *Suppose $L = \{R\}$, $\#_L(R) = 2$. Show that every team X, the domain of which contains x_i and x_j is of type $Rx_ix_j \vee \neg Rx_ix_j$.*

Exercise 2 *Let $\mathcal{M} = (\mathbb{N}, +, \cdot, 0, 1)$. Which teams $X \in \mathrm{Team}(M, \{x_0, x_1\})$ are of type (a) $=\!(x_0, x_0 + x_1)$, (b) $=\!(x_0 \cdot x_0, x_1 \cdot x_1)$.*

Exercise 3 *Let L be the vocabulary $\{f, g\}$. Describe teams $X \in \mathrm{Team}(M, \{x_0, x_1, x_2\})$ of type (a) $=\!(x_0, x_1, x_2)$, (b) $=\!(x_0, x_0, x_2)$.*

Exercise 4 *Let $\mathcal{M} = (\mathbb{N}, +, \cdot, 0, 1)$ and $X_n = \{\{(0,a),(1,b)\} : 1 < a \leq n, 1 < b \leq n, a \leq b\}$. Show that X_5 is of type $=\!(x_0 + x_1, x_0 \cdot x_1, x_0)$. This is also true for X_n for any n, but is slightly harder to prove.*

Exercise 5 *For which of the following formulas ϕ it is true that for all $X \neq \emptyset$: $\mathcal{M} \models_X \neg\phi \iff \mathcal{M} \not\models_X \phi$:*

$$(=\!(x_0, x_1) \wedge \neg x_0 = x_1)$$
$$(=\!(x_0, x_1) \to x_0 = x_1)$$
$$(=\!(x_0, x_1) \vee \neg x_0 = x_1)$$

Exercise 6 ([12]) *This exercise shows that the Closure Test is the best we can do. Let L be the vocabulary of one n-ary predicate symbol R. Let $\mathcal{M}$ be a finite set and $m \in \mathbb{N}$. Suppose S is a set of assignments of M with domain $\{x_1, ..., x_m\}$ such that S is closed under subsets. Find an interpretation $R^{\mathcal{M}} \subseteq M^n$ and a formula ϕ of $\mathcal{D}$ such that a team X with domain $\{x_1, ..., x_k\}$ is of type ϕ in $\mathcal{M}$ if and only if $X \in S$.*

Exercise 7 *Use the method of [3], mutatis mutandis, to show that there is no compositional semantics for dependence logic in which the meanings of formulas are sets of assignments (rather than sets of teams) and which agrees with Definition 5 for sentences.*

1.3 Logical Equivalence

The concept of logical consequence and the derived concept of logical equivalence are both defined below in a semantic form. In first order logic there is also a proof theoretic (or syntactic) concept of logical consequence and it coincides with the semantic concept. This fact is referred to as the Gödel Completeness Theorem. In dependence logic we have only semantic notions. There are obvious candidates for syntactic concepts but they are not well understood yet. For example, it is known that the Gödel Completeness Theorem fails badly (see Section 2.5).

Definition 12 *ψ is a logical consequence of ϕ, $\phi \Rightarrow \psi$, if for all $\mathcal{M}$ and all X with $\mathrm{dom}(X) \supseteq \mathrm{Fr}(\phi) \cup \mathrm{Fr}(\psi)$ and $\mathcal{M} \models_X \phi$ we have $\mathcal{M} \models_X \psi$. ψ is a strong logical consequence of ϕ,$\phi \Rightarrow^* \psi$, if for all $\mathcal{M}$ and for all X with $\mathrm{dom}(X) \supseteq \mathrm{Fr}(\phi) \cup \mathrm{Fr}(\psi)$ and $\mathcal{M} \models_X \phi$ we have $\mathcal{M} \models_X \psi$, and all X with $\mathrm{dom}(X) \supseteq \mathrm{Fr}(\phi) \cup \mathrm{Fr}(\psi)$ and $\mathcal{M} \models_X \neg\psi$ we have $\mathcal{M} \models_X \neg\phi$. ψ is* logically equivalent *with ϕ, $\phi \equiv \psi$, if $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$. ψ is* strongly logically equivalent *with ϕ, $\phi \equiv^* \psi$, if $\phi \Rightarrow^* \psi$ and $\psi \Rightarrow^* \phi$.*

Note that $\phi \Rightarrow^* \psi$ if and only if $\phi \Rightarrow \psi$ and $\neg\psi \Rightarrow \neg\phi$. Thus the fundamental concept is $\phi \Rightarrow \psi$ and $\phi \Rightarrow^* \psi$ reduces to it. Note also that ϕ and ψ are logically equivalent if and only if for all X with $\mathrm{dom}(X) \supseteq \mathrm{Fr}(\phi) \cup \mathrm{Fr}(\psi)$ $(\phi, X, 1) \in \mathcal{T}$ if and only if $(\psi, X, 1) \in \mathcal{T}$, and ϕ and ψ are strongly logically equivalent if and only if for all X with $\mathrm{dom}(X) \supseteq \mathrm{Fr}(\phi) \cup \mathrm{Fr}(\psi)$ and all d, $(\phi, X, d) \in \mathcal{T}$ if and only if $(\psi, X, d) \in \mathcal{T}$.

We have some familiar looking strong logical equivalences in propositional logic, reminiscent of axioms of semigroups with identity. In the following lemma we group the equivalences according to duality:

Lemma 13 *The following strong logical equivalences hold in dependence logic:*

1. $\neg\neg\phi \equiv^* \phi$

2.(a) $(\phi \wedge \top) \equiv^* \phi$

   (b) $(\phi \vee \top) \equiv^* \top$

3.(a) $(\phi \wedge \psi) \equiv^* (\psi \wedge \phi)$

   (b) $(\phi \vee \psi) \equiv^* (\psi \vee \phi)$

4.(a) $(\phi \wedge \psi) \wedge \theta \equiv^* \phi \wedge (\psi \wedge \theta)$

   (b) $(\phi \vee \psi) \vee \theta \equiv^* \phi \vee (\psi \vee \theta)$

5.(a) $\neg(\phi \vee \psi) \equiv^* (\neg\phi \wedge \neg\psi)$

   (b) $\neg(\phi \wedge \psi) \equiv^* (\neg\phi \vee \neg\psi)$

Proof. We prove only Claim (iii) (b) and leave the rest to the reader. By (E8), $(\phi\vee\psi, X, 0) \in \mathcal{T}$ if and only if $((\phi, X, 0) \in \mathcal{T}$ and $(\psi, X, 0) \in \mathcal{T})$ if and only if $(\psi \vee \phi, X, 0) \in \mathcal{T}$. Suppose then $(\phi \vee \psi, X, 1) \in \mathcal{T}$. By (E7) there are Y and Z such that $X = Y \cup Z$, $(\phi, Y, 1) \in \mathcal{T}$ and $(\psi, Z, 1) \in \mathcal{T}$. By (D7), $(\psi\vee\phi, X, 1) \in \mathcal{T}$. Conversely, if $(\psi\vee\phi, X, 1) \in \mathcal{T}$, then there are Y and Z such that $X = Y \cup Z$, $(\psi, Y, 1) \in \mathcal{T}$ and $(\phi, Z, 1) \in \mathcal{T}$. By (D7), $(\phi \vee \psi, X, 1) \in \mathcal{T}$.□

However, many familiar propositional equivalences fail on the level of strong equivalence, in particular the Law of Excluded Middle, weakening laws, and distributivity laws. See Exercise 8.

We have also some familiar looking strong logical equivalences for quantifiers. In the following lemma we again group the equivalences according to duality:

Lemma 14 *The following strong logical equivalences and consequences hold in dependence logic:*

1.(a) $\exists x_m \exists x_n \phi \equiv^* \exists x_n \exists x_m \phi$

(b) $\forall x_m \forall x_n \phi \equiv^* \forall x_n \forall x_m \phi$

2.(a) $\exists x_n(\phi \vee \psi) \equiv^* (\exists x_n \phi \vee \exists x_n \phi)$

(b) $\forall x_n(\phi \wedge \psi) \equiv^* (\forall x_n \phi \wedge \forall x_n \phi)$

3.(a) $\neg \exists x_n \phi \equiv^* \forall x_n \neg \phi$

(b) $\neg \forall x_n \phi \equiv^* \exists x_n \neg \phi$

4.(a) $\phi \Rightarrow^* \exists x_n \phi$

(b) $\forall x_n \phi \Rightarrow^* \phi$

A useful method for proving logical equivalences is the method of substitution. In first order logic this is based on the strong compositionality[2] of the semantics. The same is true for dependence logic.

Definition 15 *Suppose θ is a formula in the vocabulary $L \cup \{P\}$, where P is an n-ary predicate symbol. Let* $\mathrm{Sb}(\theta, P, \phi(x_1, ..., x_n))$ *be obtained from θ by replacing $Pt_1...t_n$ everywhere by $\phi(t_1, ..., t_n)$.*

Lemma 16 (Preservation of equivalence under substitution) *Suppose $\phi_0(x_1, ..., x_n)$ and $\phi_1(x_1, ..., x_n)$ are L-formulas of dependence logic such that $\phi_0(x_1, ..., x_n) \equiv^* \phi_1(x_1, ..., x_n)$. Suppose θ is a formula in the vocabulary $L \cup \{P\}$, where P is an n-ary predicate symbol. Then* $\mathrm{Sb}(\theta, P, \phi_0(x_1, ..., x_n)) \equiv^* \mathrm{Sb}(\theta, P, \phi_1(x_1, ..., x_n))$.

Proof. The proof is straightforward. We use induction on θ. Let us use $\mathrm{Sb}_d(\theta)$ as a shorthand for $\mathrm{Sb}(\theta, P, \phi_d)$.

Atomic case. Suppose θ is $Rt_1...t_n$. Now $\mathrm{Sb}_d(\theta) = \phi_d$. The claim follows from $\phi_0 \equiv^* \phi_1$.

Disjunction. Note that $\mathrm{Sb}_d(\phi \vee \psi) = \mathrm{Sb}_d(\phi) \vee \mathrm{Sb}_d(\psi)$. Now $(\mathrm{Sb}_d(\phi \vee \psi), X, 1) \in \mathcal{T}$ if and only if $(\mathrm{Sb}_d(\phi) \vee \mathrm{Sb}_d(\psi), X, 1) \in \mathcal{T}$ if and only if ($X = Y \cup Z$ such that $(\mathrm{Sb}_d(\phi), Y, 1) \in \mathcal{T}$ and $(\mathrm{Sb}_d(\psi), Z, 1) \in \mathcal{T}$).

[2] In compositional semantics, roughly speaking, the meaning of a compound formula is completely determined by the way the formula is built from parts and by the meanings of the parts.

By the induction hypothesis this is equivalent to ($X = Y \cup Z$ such that $(\mathrm{Sb}_{1-d}(\phi), Y, 1) \in \mathcal{T}$ and $(\mathrm{Sb}_{1-d}(\psi), Z, 1) \in \mathcal{T}$) i.e. $(\mathrm{Sb}_{1-d}(\phi) \vee \mathrm{Sb}_{1-d}(\psi), X, 1) \in \mathcal{T}$, which is finally equivalent to $(\mathrm{Sb}_{1-d}(\phi \vee \psi), X, 1) \in \mathcal{T}$. On the other hand, $(\mathrm{Sb}_d(\phi \vee \psi), X, 0) \in \mathcal{T}$ if and only if $(\mathrm{Sb}_d(\phi) \vee \mathrm{Sb}_d(\psi), X, 0) \in \mathcal{T}$ if and only if ($(\mathrm{Sb}_d(\phi), X, 0) \in \mathcal{T}$ and $(\mathrm{Sb}_d(\psi), X, 0) \in \mathcal{T}$). By the induction hypothesis this is equivalent to $(\mathrm{Sb}_{1-d}(\phi), X, 0) \in \mathcal{T}$ and $(\mathrm{Sb}_{1-d}(\psi), X, 0) \in \mathcal{T}$) i.e. $(\mathrm{Sb}_{1-d}(\phi) \vee \mathrm{Sb}_{1-d}(\psi), X, 0) \in \mathcal{T}$, which is finally equivalent to $(\mathrm{Sb}_{1-d}(\phi \vee \psi), X, 0) \in \mathcal{T}$.

Negation. $\mathrm{Sb}_e(\neg\phi) = \neg \mathrm{Sb}_e(\phi)$. Now $(\mathrm{Sb}_e(\neg\phi), X, d) \in \mathcal{T}$ if and only if $(\neg \mathrm{Sb}_e(\phi), X, d) \in \mathcal{T}$, which is equivalent to $(\mathrm{Sb}_e(\phi), X, 1-d) \in \mathcal{T}$. By the induction hypothesis this is equivalent to $(\mathrm{Sb}_{1-e}(\phi), X, 1-d) \in \mathcal{T}$ i.e. $(\neg \mathrm{Sb}_{1-e}(\phi), X, d) \in \mathcal{T}$, and finally this is equivalent to $(\mathrm{Sb}_{1-e}(\neg\phi), X, d) \in \mathcal{T}$.

Existential quantification. Note that $\mathrm{Sb}_d(\exists x_n \phi) = \exists x_n \mathrm{Sb}_d(\phi)$. We may infer, as above, that $(\mathrm{Sb}_d(\exists x_n \phi), X, 1) \in \mathcal{T}$ if and only if $(\exists x_n \mathrm{Sb}_d(\phi), X, 1) \in \mathcal{T}$ if and only if: there is $F : X \longrightarrow M$ such that $(\mathrm{Sb}_d(\phi), X(F/x_n), 1) \in \mathcal{T}$. By the induction hypothesis this is equivalent to: there is $F : X \longrightarrow M$ such that $(\mathrm{Sb}_{1-d}(\phi), X(F/x_n), 1) \in \mathcal{T}$, i.e. to $(\exists x_n \mathrm{Sb}_{1-d}(\phi), X, 1) \in \mathcal{T}$, which is finally equivalent to $(\mathrm{Sb}_{1-d}(\exists x_n \phi), X, 1) \in \mathcal{T}$. On the other hand, $(\mathrm{Sb}_d(\exists x_n \phi), X, 0) \in \mathcal{T}$ if and only if $(\exists x_n \mathrm{Sb}_d(\phi), X, 0) \in \mathcal{T}$, if and only if $((\mathrm{Sb}_d(\phi), X(M/x_n), 0) \in \mathcal{T}$. By the induction hypothesis this is equivalent to $(\mathrm{Sb}_{1-d}(\phi), X(M/x_n), 0) \in \mathcal{T}$, i.e. $(\exists x_n \mathrm{Sb}_{1-d}(\phi), X, 0) \in \mathcal{T}$, which is finally equivalent to $(\mathrm{Sb}_{1-d}(\exists x_n \phi), X, 0) \in \mathcal{T}$. $\square$

We will see later (see Section 5.3) that there is no hope of explaining $\phi \Rightarrow \psi$ in terms of some simple rules. There are examples of ϕ and ψ such that to decide whether $\phi \Rightarrow \psi$ or not, one has to decide whether there are measurable cardinals in the set theoretic universe. Likewise, there are examples of ϕ and ψ such that to decide whether $\phi \Rightarrow \psi$, one has to decide whether the Continuum Hypothesis holds.

We examine next some elementary logical properties of formulas of dependence logic. The following lemma shows that the truth of a formula depends only on the interpretations of the variables occurring free in the formula. To this end, we define $X\restriction V$ to be $\{s\restriction V : s \in X\}$.

Lemma 17 *Suppose* $V \supseteq \mathrm{Fr}(\phi)$*. Then* $\mathcal{M} \models_X \phi$ *if and only if* $\mathcal{M} \models_{X\restriction V} \phi$*.*

Proof. Key to this result is the fact that $t^{\mathcal{M}}\langle s\rangle = t^{\mathcal{M}}\langle s\restriction V\rangle$ whenever $\mathrm{Fr}(t) \subseteq V$. We use induction on ϕ to prove $(\phi, X, d) \in \mathcal{T} \iff (\phi, X\restriction V, d) \in \mathcal{T}$ whenever $\mathrm{Fr}(\phi) \subseteq V$. If ϕ is atomic, the claim is obvious, even in the case of $=(t_1, ..., t_n)$.

Disjunction. Suppose $(\phi \vee \psi, X, 1) \in \mathcal{T}$. Then $X = Y \cup Z$ such that $(\phi, Y, 1) \in \mathcal{T}$ and $(\psi, Z, 1) \in \mathcal{T}$. By the induction hypothesis $(\phi, Y\restriction V, 1) \in \mathcal{T}$ and $(\psi, Z\restriction V, 1) \in \mathcal{T}$. Of course, $X \restriction V = (Y \restriction V) \cup (Z \restriction V)$. Thus $(\phi \vee \psi, X\restriction V, 1) \in \mathcal{T}$. Conversely suppose $(\phi \vee \psi, X\restriction V, 1) \in \mathcal{T}$. Then $X \restriction V = Y \cup Z$ such that $(\phi, Y, 1) \in \mathcal{T}$ and $(\phi, Z, 1) \in \mathcal{T}$. Choose Y' and Z' such that $Y'\restriction V = Y$, $Z'\restriction V = Z$. and $X = Y' \cup Z'$. Now we have $(\phi, Y', 1) \in \mathcal{T}$ and $(\psi, Z', 1) \in \mathcal{T}$ by the induction hypothesis. Thus $(\phi \vee \psi, X, 1) \in \mathcal{T}$. Suppose then $(\phi \vee \psi, X, 0) \in \mathcal{T}$. Then $(\phi, X, 0) \in \mathcal{T}$ and $(\psi, X, 0) \in \mathcal{T}$. By the induction hypothesis $(\phi, X\restriction V, 0) \in \mathcal{T}$ and $(\psi, X\restriction V, 0) \in \mathcal{T}$. Thus $(\phi \vee \psi, X\restriction V, 0) \in \mathcal{T}$. Conversely, suppose $(\phi \vee \psi, X\restriction V, 0) \in \mathcal{T}$. Then $(\phi, X\restriction V, 0) \in \mathcal{T}$ and $(\psi, X\restriction V, 0) \in \mathcal{T}$. Now $(\phi, X, 0) \in \mathcal{T}$ and $(\psi, X, 0) \in \mathcal{T}$ by the induction hypothesis. Thus $(\phi \vee \psi, X, 0) \in \mathcal{T}$.

Negation. Suppose $(\neg\phi, X, d) \in \mathcal{T}$. Then $(\phi, X, 1-d) \in \mathcal{T}$. By the induction hypothesis $(\phi, X\restriction V, 1-d) \in \mathcal{T}$. Thus $(\neg\phi, X\restriction V, d) \in \mathcal{T}$. Conversely, suppose $(\neg\phi, X\restriction V, d) \in \mathcal{T}$. Then $(\phi, X\restriction V, 1-d) \in \mathcal{T}$. Now we have $(\phi, X, 1-d) \in \mathcal{T}$ by the induction hypothesis. Thus $(\neg\phi, X, d) \in \mathcal{T}$.

Existential quantification. Suppose $(\exists x_n, X, 1) \in \mathcal{T}$. Then there is $F : X \to M$ such that $(\phi, X(F/x_n), 1) \in \mathcal{T}$. By the induction hypothesis $(\phi, X(F/x_n)\restriction W, 1) \in \mathcal{T}$, where $W = V \cup \{n\}$. Note that, $X(F/x_n)\restriction W = (X\restriction V)(F/x_n)$. Thus $(\exists x_n\phi, X\restriction V, 1) \in \mathcal{T}$. Conversely, suppose $(\exists x_n \phi, X\lceil V, 1) \in \mathcal{T}$. Then there is $F : X\lceil V \to M$ such that $(\phi, (X\lceil V)(F/x_n), 1) \in \mathcal{T}$. Again, $X(F/x_n)\lceil W = (X\lceil V)(F/x_n)$, thus by the induction hypothesis, $(\phi, X(F/x_n), 1) \in \mathcal{T}$, i.e. $(\exists x_n \phi, X, 1) \in \mathcal{T}$.

$\square$

In the next Lemma we have the restriction, familiar from substitution rules of first order logic, that in substitution no free occurrence of a variable should become a bound.

Lemma 18 (Change of free variables) *Let the free variables of ϕ and ψ be $x_1, ..., x_n$. Let $i_1, ..., i_n$ be distinct. Let ϕ' be obtained from ϕ by replacing x_j everywhere by x_{i_j}, where $j = 1, ..., n$. If X is an assignment set with $\mathrm{dom}(X) = \{1, ..., n\}$, let X' consist of the assignments $x_{i_j} \mapsto s(x_j)$, where $s \in X$. Then $\mathcal{M} \models_X \phi \iff \mathcal{M} \models_{X'} \phi'$.*

Finally, we note the important fact that types are preserved by isomorphisms:

Lemma 19 (Isomorphism preserves truth) *Suppose $\mathcal{M} \cong \mathcal{M}'$. If $\phi \in \mathcal{D}$, then $\mathcal{M} \models \phi \iff \mathcal{M}' \models \phi$.*

Exercise 8 *Prove the following non-equivalences :*

1. (a) $\phi \vee \neg\phi \not\equiv^* \top$
   (b) $\phi \wedge \neg\phi \not\equiv^* \neg\top$, *but* $\phi \wedge \neg\phi \equiv \neg\top$
2. $(\phi \wedge \phi) \not\equiv^* \phi$, *but* $(\phi \wedge \phi) \equiv \phi$
3. $(\phi \vee \phi) \not\equiv^* \phi$
4. $(\phi \vee \psi) \wedge \theta \not\equiv^* (\phi \wedge \theta) \vee (\psi \wedge \theta)$
5. $(\phi \wedge \psi) \vee \theta \not\equiv^* (\phi \vee \theta) \wedge (\psi \vee \theta)$

Note that each of these non-equivalences is actually an equivalence in first order logic.

1.4 First Order Formulas

Some formulas of dependence logic can be immediately recognized as first order by their mere appearance. They simply do not have any occurrences of the dependence formulas $=(t_1, ..., t_n)$ as subformulas. We then appropriately call them *first order*. Other formulas may be apparently non-first order, but turn out to be logically equivalent to a first order formula. Our goal in this section is to show that for apparently first order formulas our dependence logic truth definition (Definition 5 with $X \neq \emptyset$) coincides with the standard first order truth definition (Definition **??**). We also give a simple criterion called the *Flatness Test* that can be used to test whether a formula of dependence logic is logically equivalent to a first order formula.

We begin by proving that a team is of a first order type ϕ if every assignment s in X satisfies ϕ. Note the a priori difference between an assignment s satisfying a first order formula ϕ and the team $\{s\}$ being of type ϕ. We will show that these conditions are equivalent, but this indeed needs a proof.

Proposition 20 *If an L-formula ϕ of dependence logic is first order, then:*

1. *If $\mathcal{M} \models_s \phi$ for all $s \in X$, then $(\phi, X, 1) \in \mathcal{T}$.*
2. *If $\mathcal{M} \models_s \neg\phi$ for all $s \in X$, then $(\phi, X, 0) \in \mathcal{T}$.*

Proof. We use induction:

1. If $t_1^{\mathcal{M}}\langle s\rangle = t_2^{\mathcal{M}}\langle s\rangle$ for all $s \in X$, then $(t_1 = t_2, X, 1) \in \mathcal{T}$ by D1.

2. If $t_1^{\mathcal{M}}\langle s\rangle \neq t_2^{\mathcal{M}}\langle s\rangle$ for all $s \in X$, then $(t_1 = t_2, X, 0) \in \mathcal{T}$ by D2.

3. $(=(), X, 1) \in \mathcal{T}$ by D3.

4. $(=(), \emptyset, 0) \in \mathcal{T}$ by D4.

5. If $(t_1^{\mathcal{M}}\langle s\rangle, ..., t_n^{\mathcal{M}}\langle s\rangle) \in R^{\mathcal{M}}$ for all $s \in X$, then $(Rt_1...t_n, X, 1) \in \mathcal{T}$ by D5.

6. If $(t_1^{\mathcal{M}}\langle s\rangle, ..., t_n^{\mathcal{M}}\langle s\rangle) \notin R^{\mathcal{M}}$ for all $s \in X$, then $(Rt_1...t_n, X, 0) \in \mathcal{T}$ by D6.

7. If $\mathcal{M} \models_s \neg(\phi \vee \psi)$ for all $s \in X$, then $\mathcal{M} \models_s \neg\phi$ for all $s \in X$ and $\mathcal{M} \models_s \neg\psi$ for all $s \in X$, whence $(\phi, X, 0) \in \mathcal{T}$ and $(\psi, X, 0) \in \mathcal{T}$, and finally $((\phi \vee \psi), X, 0) \in \mathcal{T}$ by D7.

8. If $\mathcal{M} \models_s \phi \vee \psi$ for all $s \in X$, then $X = Y \cup Z$ such that $\mathcal{M} \models \phi$ for all $s \in Y$ and $\mathcal{M} \models \psi$ for all $s \in Z$. Thus $(\psi, Y, 1) \in \mathcal{T}$ and $(\psi, Z, 1) \in \mathcal{T}$, whence $((\phi \vee \psi), Y \cup Z, 1) \in \mathcal{T}$ by D8.

9. If $\mathcal{M} \models_s \neg\phi$ for all $s \in X$, then $(\phi, X, 0) \in \mathcal{T}$, whence $(\neg\phi, X, 1) \in \mathcal{T}$ by D9.

10. If $\mathcal{M} \models_s \neg\neg\phi$ for all $s \in x$, then $(\phi, X, 1) \in \mathcal{T}$, whence $(\neg\phi, X, 0) \in \mathcal{T}$ by D10.

11. If $\mathcal{M} \models_s \exists x_n\phi$ for all $s \in X$, then for all $s \in X$ there is $a_s \in M$ such that $\mathcal{M} \models_{s(a_s/x_n)} \phi$. Now $(\phi, \{s\}(F/x_n), 1) \in \mathcal{T}$ for $F : X \to M$ such that $F(s) = a_s$. Thus $(\exists x_n\phi, X, 1) \in \mathcal{T}$.

12. If $\mathcal{M} \models_s \neg\exists x_n\phi$ for all $s \in X$, then for all $a \in M$ we have for all $s \in X$ $\mathcal{M} \models_{s(a/x_n)} \neg\phi$. Now $(\phi, X(M/x_n), 0) \in \mathcal{T}$. Thus $(\exists x_n\phi, X, 0) \in \mathcal{T}$.

$\square$

Now for the other direction:

Proposition 21 *If an L-formula ϕ of dependence logic is first order, then:*

1. *If $(\phi, X, 1) \in \mathcal{T}$, then $\mathcal{M} \models_s \phi$ for all $s \in X$.*
2. *If $(\phi, X, 0) \in \mathcal{T}$, then $\mathcal{M} \models_s \neg\phi$ for all $s \in X$.*

Proof. We use induction:

1. If $(t_1 = t_2, X, 1) \in \mathcal{T}$, then $t_1^{\mathcal{M}}\langle s\rangle = t_2^{\mathcal{M}}\langle s\rangle$ for all $s \in X$ by E1.

2. If $(t_1 = t_2, X, 0) \in \mathcal{T}$, then $t_1^{\mathcal{M}}\langle s\rangle \neq t_2^{\mathcal{M}}\langle s\rangle$ for all $s \in X$ by E2.

3. $(=(), X, 1) \in \mathcal{T}$ and likewise $\mathcal{M} \models_s \top$ for all $s \in X$.

4. $(=(), \emptyset, 0) \in \mathcal{T}$ and likewise $\mathcal{M} \models_s \neg\top$ for all (i.e. none) $s \in \emptyset$.

5. If $(Rt_1...t_n, X, 1) \in \mathcal{T}$, then $(t_1^{\mathcal{M}}\langle s\rangle, ..., t_n^{\mathcal{M}}\langle s\rangle) \in R^{\mathcal{M}}$ for all $s \in X$ by E5.

6. If $(Rt_1...t_n, X, 0) \in \mathcal{T}$, then $(t_1^{\mathcal{M}}\langle s\rangle, ..., t_n^{\mathcal{M}}\langle s\rangle) \notin R^{\mathcal{M}}$ for all $s \in X$ by E6.

7. If $(\phi \vee \psi, X, 0) \in \mathcal{T}$, then $(\phi, X, 0) \in \mathcal{T}$ and $(\psi, X, 0) \in \mathcal{T}$ by E7, whence $\mathcal{M} \models_s \neg\phi$ for all $s \in X$ and $\mathcal{M} \models_s \neg\psi$ for all $s \in X$, whence finally $\mathcal{M} \models_s \neg(\phi \vee \psi)$ for all $s \in X$.

8. If $(\phi \vee \psi, X, 1) \in \mathcal{T}$, then $X = Y \cup Z$ such that $(\phi, Y, 1) \in \mathcal{T}$ and $(\psi, Z, 1) \in \mathcal{T}$ by E8, whence $\mathcal{M} \models_s \phi$ for all $s \in Y$ and $\mathcal{M} \models_s \psi$ for all $s \in Z$, and therefore $\mathcal{M} \models_s \phi \wedge \psi$ for all $s \in X$.

We leave the other cases as an exercise. □

We are now ready to combine Propositions 20 and 21 in order to prove that the semantics we gave in Definition 5 coincides in the case of first order formulas with the more traditional semantics given in Section **??**.

Corollary 22 *Let ϕ be a first order L-formula of dependence logic. Then:*

1. $\mathcal{M} \models_{\{s\}} \phi$ *if and only if* $\mathcal{M} \models_s \phi$.
2. $\mathcal{M} \models_X \phi$ *if and only if* $\mathcal{M} \models_s \phi$ *for all* $s \in X$.

Proof. If $\mathcal{M} \models_{\{s\}} \phi$, then $\mathcal{M} \models_s \phi$ by Proposition 21. If $\mathcal{M} \models_s \phi$, then $\mathcal{M} \models_{\{s\}} \phi$ by Proposition 20.□

We shall now introduce a test, comparable to the Closure Test introduced above. The Closure Test was used to test which types of teams are definable in dependence logic. With our new test we can check whether a type is first order, at least up to logical equivalence.

Definition 23 (Flatness[3] Test) *We say that ϕ passes the* Flatness Test *if for all $\mathcal{M}$ and X:$\mathcal{M} \models_X \phi \iff (\mathcal{M} \models_{\{s\}} \phi$ for all $s \in X)$.*

Proposition 24 *Passing the Flatness Test is preserved by logical equivalence.*

Proof. Suppose $\phi \equiv \psi$ and ϕ passes the Flatness Test. Suppose $\mathcal{M} \models_{\{s\}} \psi$ for all $s \in X$. By logical equivalence $\mathcal{M} \models_{\{s\}} \phi$ for all $s \in X$. But ϕ passes the Flatness Test. So $\mathcal{M} \models_X \phi$, and therefore by our assumption, $\mathcal{M} \models_X \psi$. $\square$

Proposition 25 *Any L-formula ϕ of dependence logic that is logically equivalent to a first order formula satisfies the Flatness Test.*

Proof. Suppose $\phi \equiv \psi$, where ψ is first order. Since ψ satisfies the Flatness Test, also ϕ does, by Proposition 24. $\square$

Example 26 $=\!(x_0, x_1)$ *does not pass the Flatness Test, as the team $X = \{\{(0,0),(1,1)\},\{(0,1),(1,1)\}\}$ in a model $\mathcal{M}$ with at least two elements 0 and 1 shows. Namely, $\mathcal{M} \not\models_X =\!(x_0, x_1)$, but $\mathcal{M} \models_{\{s\}} =\!(x_0, x_1)$ for $s \in X$. We conclude that $=\!(x_0, x_1)$ is not logically equivalent to a first-order formula.*

Example 27 $\exists x_2(=\!(x_0, x_2) \wedge x_2 = x_1)$ *does not pass the Flatness Test, as the team $X = \{s, s'\}$, $s = \{(0,0),(1,1)\}, s' = \{(0,0),(1,0)\}$ in a model $\mathcal{M}$ with at least two elements 0 and 1 shows. Namely, if $F : X \to M$ witnesses $\mathcal{M} \models_{X(F/x_2)} =\!(x_0, x_2) \wedge x_2 = x_1$, then $s(x_0) = s'(x_0)$, but $1 = s(x_1) = F(s) = F(s') = s'(x_1) = 0$, a contradiction. We conclude that $\exists x_2(=\!(x_0, x_2) \wedge x_2 = x_1)$ is not logically equivalent to a first-order formula.*

Example 28 *Let $L = \{+, \cdot, 0, 1, <\}$ and $\mathcal{M} = (\mathbb{N}, +, \cdot, 0, 1, <)$, the standard model of arithmetic. The formula $\exists x_0(=\!(x_0) \wedge (x_1 < x_0))$ fails to meet the Flatness Test. To see this, we first note that if*

$X = \{s\}$, then X is of the type of the formula, as we can choose a_s to be equal to $s(x_1) + 1$. On the other hand, let $X = \{s_n : n \in \mathbb{N}\}$, where $s_n(x_1) = n$. It is impossible to choose a such that $a > s_n(x_1)$ for all $n \in \mathbb{N}$.

Exercise 9 *Find a logically equivalent first order formula for $\exists x_0(=(x_1, x_0) \wedge Px_0)$.*

Exercise 10 *Which of the following formulas are logically equivalent to a first order formula: $=(x_0, x_1, x_2) \wedge x_0 = x_1$, $(=(x_0, x_2) \wedge x_0 = x_1) \rightarrow =(x_1, x_2)$, $=(x_0, x_1, x_2) \vee \neg =(x_0, x_1, x_2)$*

Exercise 11 *Let $L = \emptyset$ and let $\mathcal{M}$ be an L-structure with $M = \{0, 1\}$. Show that the following types of a team X with domain $\{x_0, x_1, x_2\}$ are non-first order:*

a) $\exists x_0(=(x_2, x_0) \wedge \neg(x_0 = x_1))$

b) $\exists x_0(=(x_2, x_0) \wedge (x_0 = x_1 \vee x_0 = x_2))$

c) $\exists x_0(=(x_2, x_0) \wedge (x_0 = x_1 \wedge \neg x_0 = x_2))$.

Exercise 12 *Let $L = \{R\}$, $\#(R) = 1$. Find an L-structure $\mathcal{M}$ which demonstrates that the following properties of a team X with domain $\{x_0, x_1, x_2\}$ are non-first order:*

a) $\exists x_0(Rx_0 \wedge =(x_1, x_0) \wedge \neg x_0 = x_2)$

b) $\exists x_0(=(x_2, x_0) \wedge (Rx_0 \leftrightarrow Rx_1))$

c) $\exists x_0(=(x_2, x_0) \wedge ((Rx_1 \wedge \neg Rx_0) \vee (\neg Rx_1 \wedge Rx_0)))$.

Exercise 13 *A formula ϕ of dependence logic is* ***coherent*** *if the following holds: Any team X is of type ϕ if and only if for every $s, s' \in X$ the pair team $\{s, s'\}$ is of type ϕ. Note that the formula $(=(x_1, ..., x_n) \wedge \phi)$ is coherent if ϕ is. Show that for every first order ϕ with $\mathrm{Fr}(\phi) = \{x_1\}$, the type $\exists x_0(=(x_1, x_0) \wedge \phi)$ is coherent. Give an example of a formula ϕ of dependence logic which is not coherent (see Exercise 13 for the definition of coherence).*

1.5 The Flattening Technique

We now introduce a technique which may seem frivolous at first sight but proves very useful in the end. This is the process of flattening, by which we mean getting rid of the dependence formulas $=(t_1, ..., t_n)$. Naturally we lose something, but this is a method to reveal whether a formula has genuine occurrences of dependence or just ersatz ones.

Definition 29 *The* flattening ϕ^f *of a formula* ϕ *of dependence logic is defined by induction as follows:*

$$\begin{array}{lcl|lcl} (t_1 = t_2)^f & = & t_1 = t_2 & (Rt_1...t_n)^f & = & Rt_1...t_n \\ (=(t_1, ..., t_n))^f & = & \top & (\neg\phi)^f & = & \neg\phi^f \\ (\phi \vee \psi)^f & = & \phi^f \vee \psi^f & (\exists x_n \phi)^f & = & \exists x_n \phi^f \end{array}$$

Note that the result of flattening is always first order. The main feature of flattening is that it preserves truth:

Proposition 30 *If* ϕ *is an* L*-formula of dependence logic, then* $\phi \Rightarrow \phi^f$*.*

Proof. Inspection of Definition 5 reveals immediately that in each case where $(\phi, X, d) \in \mathcal{T}$, we also have $(\phi^f, X, d) \in \mathcal{T}$. □

We can use the above proposition to prove various useful little results which are often comforting in enforcing our intuition. We first point out that although a team may be of the type of both a formula and its negation, this can only happen if the team is empty and thereby is of the type of any formula.

Corollary 31 $\mathcal{M} \models_X (\phi \wedge \neg\phi)$ *if and only if* $X = \emptyset$*.*

Proof. We already know that $\mathcal{M} \models_\emptyset (\phi \wedge \neg\phi)$. On the other hand, if $\mathcal{M} \models_X (\phi \wedge \neg\phi)$ and $s \in X$, then $\mathcal{M} \models_s (\phi^f \wedge \neg\phi^f)$, a contradiction. □

Corollary 32 (Modus Ponens) *Suppose* $\mathcal{M} \models_X \phi \to \psi$ *and* $\mathcal{M} \models_X \phi$. *Then* $\mathcal{M} \models_X \psi$. *(See also Exercise 15.)*

Proof. $\mathcal{M} \models_X \neg\phi \vee \psi$ implies $X = Y \cup Z$ such that $\mathcal{M} \models_Y \neg\phi$ and $\mathcal{M} \models_Z \psi$. Now $\mathcal{M} \models_Y \phi$ and $\mathcal{M} \models_Y \neg\phi$, whence $Y = \emptyset$. Thus $X = Z$ and $\mathcal{M} \models_X \psi$ follows. □

In general we may conclude from Proposition 30 that a non-empty team cannot have the type of a formula which is contradictory in first order logic when flattened. When all the subtle properties of dependence logic are laid bare in front of us, we tend to seek solace in anything solid, anything that we know for certain from our experience in first order logic. Flattening is one solace. By simply ignoring the dependence statements $=(t_1, ..., t_n)$ we can recover in a sense the first-order content of the formula. When we master this technique, we begin to understand the effect of the presence of dependence statements in a formula.

Example 33 *No non-empty team can have the type of any of the following formulas, whatever formulas of dependence logic the formulas* ϕ *and* ψ *are:*

$$\begin{array}{l} \phi(c) \wedge \forall x_0 \neg\phi(x_0), \\ \forall x_0 \neg\phi \wedge \forall x_0 \neg\psi \wedge \exists x_0(\phi \vee \psi), \\ \neg(((\phi \to \psi) \to \phi) \to \phi). \end{array}$$

The flattenings of these formulas are respectively

$$\begin{array}{l} \phi^f(c) \wedge \forall x_0 \neg\phi^f(x_0), \\ \forall x_0 \neg\phi^f \wedge \forall x_0 \neg\psi^f \wedge \exists x_0(\phi^f \vee \psi^f), \\ \neg(((\phi^f \to \psi^f) \to \phi^f) \to \phi^f), \end{array}$$

none of which can be satisfied by any assignment in first order logic. In the last case one can use truth-tables to verify this.

As the previous example shows, the Truth-Table Method, so useful in predicate calculus, has a role also in dependence logic.

Exercise 14 *Let* ϕ *be the formula* $\exists x_0 \forall x_1 \neg(=(x_2, x_1) \wedge (x_0 = x_1))$ *of* $\mathcal{D}$. *Show that the flattening of* ϕ *is not a strong logical consequence of* ϕ.

Exercise 15 *If* $\mathcal{M} \models_X (\phi \to \psi)$ *and* $\mathcal{M} \models_X \neg\psi$, *then* $\mathcal{M} \models_X \neg\phi$.

Exercise 16 *Show that no non-empty team can have the type of any of the following formulas:*

$$\neg =(x_0, x_1)$$
$$\neg(=(x_0, x_1) \to =(x_2, x_1))$$
$$\neg =(fx_0, x_0) \vee \neg =(x_0, fx_0)$$
$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 \neg(\phi \to =(x_0, x_1))$$

Exercise 17 *Explain the difference between teams of type* $=(x_0, x_2) \wedge =(x_1, x_2)$ *and teams of type* $=(x_0, x_1, x_2)$.

Exercise 18 *If* ϕ *has only* x_0 *and* x_1 *free, then* $\forall x_0 \exists x_1 \phi \Rightarrow \forall x_0 \exists x_1(=(x_0, x_1)$ $\phi)$.

Exercise 19 *Show that the formulas* $\forall x_0 \exists x_1 \forall x_2 \exists x_3(=(x_0, x_1) \wedge =(x_2, x_3) \wedge \phi)$ *and* $\forall x_2 \exists x_3 \forall x_0 \exists x_1(=(x_0, x_1) \wedge =(x_2, x_3) \wedge \phi)$, *where* ϕ *is first order, are logically equivalent.*

Exercise 20 *Prove* $\exists x_n(\phi \wedge \psi) \equiv^* \phi \wedge \exists x_n \psi$ *if* x_n *not free in* ϕ.

Exercise 21 *Prove that* $\models \exists x_1(=(x_1) \wedge x_1 = c)$ *but* $\not\models \forall x_0 \exists x_1(=(x_1) \wedge x_1 = x_0)$.

Exercise 22 (Prenex Normal Form) *A formula of dependence logic is in prenex normal form if all quantifiers are in the beginning of the formula. Use Lemma 14 and Exercise 20 to prove that every formula of dependence logic is strongly equivalent to a formula which has the same free variables and is in the prenex normal form.*

1.6 Dependence/Independence Friendly Logic

We review the relation of our dependence logic $\mathcal{D}$ to the independence friendly logics of [10], [11] and [12].

The *backslashed quantifier*

$$\exists x_n \backslash \{x_{i_0}, ..., x_{i_{m-1}}\}\phi, \tag{1.2}$$

introduced in [11], with the intuitive meaning

$$\text{“there exists } x_n \text{, depending only on } x_{i_0}...x_{i_{m-1}} \text{, such that } \phi \text{,”} \tag{1.3}$$

can be defined in dependence logic by the formula

$$\exists x_n (=(x_{i_0}, ..., x_{i_{m-1}}, x_n) \wedge \phi). \tag{1.4}$$

Conversely, we can define $=(x_{i_0}, ..., x_{i_{m-1}})$ in terms of (1.2) by means of the formula

$$\exists x_n \backslash \{x_{i_0}, ..., x_{i_{m-2}}\}(x_n = x_{i_{m-1}}). \tag{1.5}$$

Similarly, we can define $=(t_1, ..., t_n)$ in terms of (1.2), when $t_1, ..., t_n$ are terms.

Dependence friendly logic, denoted DF, is the fragment of dependence logic obtained by leaving out the atomic dependence formulas $=(t_1, ..., t_n)$ and adding all the backslashed quantifiers (1.2). Dependence logic and DF have the same expressive power, not just on the level of sentences, but even on the level of formulas in the following sense:

Proposition 34 *1. For every ϕ in $\mathcal{D}$ there is ϕ^* in DF so that for all models $\mathcal{M}$ and all teams X: $\mathcal{M} \models_X \phi \iff \mathcal{M} \models_X \phi^*$.*

2. For every ψ in DF there is ψ^{**} in $\mathcal{D}$ so that for all models $\mathcal{M}$ and all teams X: $\mathcal{M} \models_X \psi \iff \mathcal{M} \models_X \psi^{**}$.

We can base the study of dependence either on the atomic formulas $t_1 = t_n$, $Rt_1...t_n$, $=(t_1, ..., t_n)$, together with the logical operations $\neg, \vee, \exists x_n$, as we have done in this book, or on the atomic formulas $t_1 =$

t_n, $Rt_1...t_n$, together with the logical operations $\neg, \vee, \exists x_n \backslash \{x_{i_0}, ..., x_{i_{m-1}}\}$. The results of this book remain true if $\mathcal{D}$ is replaced by DF.

The *slashed quantifier*

$$\exists x_n / \{x_{i_0}, ..., x_{i_{m-1}}\}\phi, \tag{1.6}$$

used in [12] has the intuitive meaning

$$\text{“there exists } x_n\text{, independently of } x_{i_0}...x_{i_{m-1}}\text{, such that } \phi\text{,”} \tag{1.7}$$

which we take to mean

$$\begin{aligned}&\text{“there exists } x_n\text{, depending only on variables } \textbf{other} \text{ than}\\ &x_{i_0}...x_{i_{m-1}}\text{, such that } \phi\text{,”}\end{aligned} \tag{1.8}$$

If the other variables, referred to in (1.8) are $x_{j_0}...x_{j_{l-1}}$, then (1.7) is intuitively equivalent with

$$\exists x_n \backslash \{x_{j_0}, ..., x_{j_{l-1}}\}\phi. \tag{1.9}$$

Independence friendly logic, denoted IF, is the fragment of dependence logic obtained by leaving out the atomic dependence formulas $=(t_1, ..., t_n)$ and adding all the slashed quantifiers (1.6) with (1.7) (or rather (1.9)) as their meaning. Sentences of dependence logic and IF have the same expressive power in the following sense:

1. For every sentence ϕ in $\mathcal{D}$ there is a sentence ϕ^* in IF so that for all models $\mathcal{M}$: $\mathcal{M} \models \phi \iff \mathcal{M} \models \phi^*$.
2. For every sentence ψ in IF there is a sentence ψ^{**} in $\mathcal{D}$ so that for all models $\mathcal{M}$: $\mathcal{M} \models \psi \iff \mathcal{M} \models \psi^{**}$.

We observed that we can base the study of dependence on $\mathcal{D}$ or DF and everything will go through more or less in the same way. However, IF differs more from $\mathcal{D}$ than DF, even if the expressive power is in the above sense the same as that of $\mathcal{D}$, and even if there is the intuitive equivalence of (1.7) and (1.9).

Dealing with (1.8) rather than (1.3) involves the complication, that one has to decide whether “other variable” refers to other variables actually appearing in a formula ϕ, or to other variables in the domain of the team X under consideration. In the latter case variables not occurring in the formula ϕ may still determine whether the team X is of type ϕ.

Consider, for example, the formula $\theta : \exists x_0/\{x_1\}(x_0 = x_1)$. The teams

x_0	x_1	x_2
1	1	1
1	3	3
1	8	8

x_0	x_1	x_2
1	1	5
1	3	2
1	8	1

are of type θ as we can let x_0 depend on x_2. The variable x_2, which does not occur in θ, *signals* what x_1 is. However, the team

x_0	x_1	x_2
1	1	5
1	3	5
1	8	5

is not of type θ, even though all three teams agree on all variables that occur in θ. The corresponding formula $\exists x_0 \backslash \{x_2\}(x_0 = x_1)$ of DF avoids this as all variables that are actually used are mentioned in the formula. In this respect DF is easier to work with than IF.

Exercise 23 *Give a logically equivalent formula in $\mathcal{D}$ for the DF-formula $\exists x_2 \backslash R x_1 x_2$.*

Exercise 24 *Give for both of the following $\mathcal{D}$-formulas*

$$\exists x_2 \exists x_3 (=(x_0, x_2) \wedge =(x_1, x_3) \wedge R x_0 x_1 x_2 x_3)$$
$$=(x_0) \vee =(x_1)$$

a logically equivalent formula in DF.

Exercise 25 *Give for each of the following $\mathcal{D}$-sentences ϕ*

$$\forall x_0 \exists x_1 (\neg =(x_0, x_1) \wedge \neg x_1 = x_0)$$
$$\forall x_0 \forall x_1 \exists x_2 (=(x_1, x_2) \wedge \neg x_2 = x_1)$$

a sentence ϕ^* *in IF so that* ϕ *and* ϕ^* *have the same models.*

Exercise 26 *Give for both of the following IF-sentences*

$$\forall x_0 \exists x_1/\{x_0\}(x_0 = x_1)$$
$$\forall x_0 \exists x_1/\{x_0\}(x_1 \leq x_0)$$

a first order sentence with the same models.

Exercise 27 *Give a definition of* $=\!(t_1, ..., t_n)$ *in DF.*

Chapter 2

Examples

We study now some more complicated examples involving many quantifiers. In all these examples we use quantifiers to express the existence of some functions. There is a certain easy trick for accomplishing this which hopefully becomes apparent to the reader. The main idea is that some variables stand for arguments and some stand for values of functions that the sentence stipulates to exist.

2.1 Even Cardinality

On a finite set $\{a_1, ..., a_n\}$ of even size one can define a one to one function f which is its own inverse and has no fixed points, as in the picture:

Conversely, any finite set with such a function has even cardinality. In the following sentence we think of $f(x_0)$ as x_1 and of $f(x_2)$ as x_3. So x_1 depends only on x_0, and x_3 depends only on x_2, which is guaranteed by $=(x_2, x_3)$. To make sure f has no fixed points we stipulate $\neg(x_0 = x_1)$. The condition $(x_1 = x_2 \to x_3 = x_0)$ says in effect $(f(x_0) = x_2 \to f(x_2) = x_0)$. i.e. $f(f(x_0)) = x_0$. Let:

$$\begin{aligned}\Phi_{\text{even}} : \forall x_0 \exists x_1 \forall x_2 \exists x_3 (=\!(x_2, x_3) \ &\wedge\ \neg(x_0 = x_1)\\ &\wedge\ (x_0 = x_2 \rightarrow x_1 = x_3)\\ &\wedge\ (x_1 = x_2 \rightarrow x_3 = x_0))\end{aligned}$$

The sentence Φ_{even} of dependence logic is true in a finite structure if and only if the size of the structure is even.

Exercise 28 *Give a sentence of dependence logic which is true in a finite structure if and only if the size of the structure is odd. Note that* $\neg\Phi_{\text{even}}$ *would not do.*

2.2 Cardinality

The domain of a structure is infinite if and only if there is a one to one function that maps the domain into a proper subset. For example, if the domain contains an infinite set $A = \{a_0, a_1, ...\}$ we can map A onto the proper subset $\{a_1, a_2, ...\}$ with the mapping $a_n \mapsto a_{n+1}$, and the outside of A onto itself by the identity mapping. On the other hand, if $f : M \rightarrow M$ is a one to one function that not have a in its range, then $\{a, f(a), f(f(a)), ...\}$ is an infinite subset.

In the following sentence we think of $f(x_0)$ as x_1 and of $g(x_2)$ as x_3. So x_1 depends only on x_0, and x_3 depends only on x_2, which is guarantees by $=\!(x_2, x_3)$. The condition $\neg(x_1 = x_4)$ says x_4 is outside the range of the function f. To make sure that $f = g$ we stipulate $(x_0 = x_2 \rightarrow x_1 = x_3)$. The condition $(x_1 = x_3 \rightarrow x_0 = x_2)$ says f is one to one. Let:

$$\begin{aligned}\Phi_{\infty} : \exists x_4 \forall x_0 \exists x_1 \forall x_2 \exists x_3 (=\!(x_2, x_3) \ &\wedge\ \neg(x_1 = x_4)\\ &\wedge\ (x_0 = x_2 \leftrightarrow x_1 = x_3))\end{aligned}$$

We conclude that Φ_∞ is true in a structure if and only if the domain of the structure is infinite.

Exercise 29 *A* graph *is a pair $\mathcal{M} = (G, E)$ where G is a set of elements called* vertices *and E is an anti-reflexive symmetric binary relation on G called the* edge-*relation. The* degree *of a vertex is the number of vertices that are connected by a (single) edge to v. The degree of v is said to be infinite if the set of vertices that are connected by an edge to v is infinite. Give a sentence of dependence logic which is true in a graph if and only if every vertex has infinite degree.*

Exercise 30 *Give a sentence of dependence logic which is true in a graph if and only if the graph has infinitely many isolated vertices (a vertex is* isolated *if it has no neighbors).*

Exercise 31 *Give a sentence of dependence logic which is true in a graph if and only if the graph has infinitely many vertices of infinite degree (the* degree *of a vertex is the cardinality of the set of neighbor of the vertex).*

A more general question about cardinality is *equicardinality*. In this case we have two unary predicates P and Q on a set M and we want to know whether they have the same cardinality; that is, whether there is a bijection f from P to Q. In the following sentence $\Phi_=$ we think of $f(x_0)$ as x_1 and of $f(x_2)$ as x_3.

$$\begin{aligned}\Phi_= : \forall x_0 \exists x_1 \forall x_2 \exists x_3 (=\!(x_2, x_3) \ \wedge \ & ((Px_0 \wedge Qx_2) \rightarrow \\ & (Qx_1 \wedge Px_3 \wedge \\ & (x_0 = x_3 \leftrightarrow x_1 = x_2))))\end{aligned}$$

Suppose we want to test whether a unary predicate Q has at least as many elements as another unary predicate P. Here we can use a simplification of $\Phi_=$:

$$\Phi_{\leq} : \forall x_0 \exists x_1 \forall x_2 \exists x_3 (=(x_2, x_3) \ \wedge \ (Px_0 \to (Qx_1 \wedge (x_0 = x_2 \leftrightarrow x_1 = x_3))))$$

On the other hand, the following variant of $\Phi_=$ clearly expresses the isomorphism of two linear orders $(P, <_P)$ and $(Q, <_Q)$:

$$\begin{aligned}\Phi_{\cong} : \forall x_0 \exists x_1 \forall x_2 \exists x_3 (=(x_2, x_3) \ & \wedge \ ((Px_0 \wedge Qx_2) \to \\ & \wedge \ (Qx_1 \wedge Px_3 \wedge \\ & \wedge \ (x_0 <_P x_3 \leftrightarrow x_1 <_Q x_2))))\end{aligned}$$

An isomorphism $\mathcal{M} \to \mathcal{M}$ is called an *automorphism*. The identity mapping is, of course, always an automorphism. An automorphism is *non-trivial* if it is not the identity mapping. Below is a picture of a finite structure with a non-trivial automorphism:

A structure is *rigid* if it has only one automorphism, namely the identity. Finite linear orders and e.g. $(\mathbb{N}, <)$ are rigid[1], but for example $(\mathbb{Z}, <)$ is non-rigid. We can express the non-rigidity of a linear order with the following sentence

$$\begin{aligned}\Phi_{\mathrm{nr}} : \exists x_4 \forall x_0 \exists x_1 \forall x_2 \exists x_3 (=(x_2, x_3) \ & \wedge \ (x_0 = x_4 \to \neg(x_1 = x_4)) \\ & \wedge \ (x_0 < x_3 \leftrightarrow x_1 < x_2))\end{aligned}$$

Exercise 32 *Write down a sentence of $\mathcal{D}$ which is true in a group*[2] *if and only if the group is non-rigid.*

Exercise 33 *Write down a sentence of $\mathcal{D}$ which is true in a finite structure $\mathcal{M}$ if and only if the unary predicate P contains in $\mathcal{M}$ at least half of the elements of M.*

[1] Any automorphism has to map the first element to the first element, the second element to the second element, the third element to the third element, etc

[2] A group is a structure $(G, \circ, e)$ with a binary function $\circ$ and a constant e such that (1) for all $a, b, c \in G$: $(a \circ b) \circ c = a \circ (b \circ c)$, (2) for all $a \in G$: $e \circ a = a \circ e = a$, (3) for all $a \in G$ there is $b \in G$ such that $a \circ b = b \circ a = e$. The group is *abelian* if in addition: (4) for all $a, b \in G$: $a \circ b = b \circ a$.

Exercise 34 *A natural number n is a prime power if and only if there is a finite field of n elements. Use this fact to write down a sentence of $\mathcal{D}$ in the empty vocabulary which has finite models of exactly prime power cardinalities.*

Exercise 35 *A group $(G, \circ, e)$ is* right orderable *if there is a partial order $\leq$ in the set G such that $x \leq y$ implies $x \circ z \leq y \circ z$ for all x, y, z in G. Write down a sentence of $\mathcal{D}$ which is true in a group if and only if the group is right orderable.*

Exercise 36 *An abelian group $(G, +, 0)$ is the additive group of a field if there are a binary operation $\cdot$ on G and an element 1 in G such that $(G, +, \cdot, 0, 1)$ is a field. Write down a sentence of $\mathcal{D}$ which is true in an abelian group if and only if the group is the additive group of a field.*

2.3 Completeness

Suppose we want to test whether a linear order $<$ on a set M is *complete* or not, i.e. whether every non-empty $A \subseteq M$ with an upper bound has a least upper bound. Since we have to talk about arbitrary subsets A of a domain M, we use a technique called *guessing*. This is nothing else than fixing an element a of M and then taking an arbitrary function from M to M. We call a the "head" as if we were tossing coin. The set A corresponds to the set of elements of M mapped to the head. For simplicity, we take the head to be an upper bound of A which we assume to exist anyway.

A linear order is incomplete if and only if there is a non-empty initial segment A without a last point but with an upper bound such that for every element not in A there is a smaller element not in A. To express this we use the sentence

$$
\begin{aligned}
\Phi_{\mathrm{cmpl}} : \exists x_6 \exists x_7 \forall x_0 \exists x_1 \forall x_2 \exists x_3 \\
\forall x_4 \exists x_5 \forall x_8 \exists x_9 (\quad & =\!(x_2, x_3) \wedge =\!(x_4, x_5) \wedge =\!(x_8, x_9) \\
\wedge \; & (x_0 = x_2 \rightarrow x_1 = x_3) \\
\wedge \; & (x_1 = x_6 \rightarrow x_0 < x_6) \\
\wedge \; & (x_0 = x_7 \rightarrow x_1 = x_6) \\
\wedge \; & ((x_0 < x_2 \wedge x_3 = x_6) \\
& \rightarrow x_1 = x_6) \\
\wedge \; & ((\neg(x_1 = x_6) \wedge x_0 = x_4 \wedge x_2 = x_5) \\
& \rightarrow (x_5 < x_0 \wedge \neg(x_3 = x_6))) \\
\wedge \; & ((x_0 = x_8 \wedge x_1 = x_6 \wedge x_2 = x_9) \\
& \rightarrow (x_8 < x_9 \wedge x_3 = x_6)))
\end{aligned}
$$

The sentence Φ_{cmpl} is true in a linear order if and only if the linear order is incomplete. (Φ_{cmpl} is not necessarily the simplest one with this property.)

Explanation of Φ_{cmpl}: The mapping $x_0 \mapsto x_1$ is the guessing function and $x_2 \mapsto x_3$ is a copy of it, as witnessed by $(x_0 = x_2 \rightarrow x_1 = x_3)$. x_6 is the head, therefore we have $(x_1 = x_6 \rightarrow x_0 < x_6)$. x_7 manifests non-emptiness of the guessed initial segment as witnessed by $(x_0 = x_7 \rightarrow x_1 = x_6)$. The clause $((x_0 < x_2 \wedge x_3 = x_6) \rightarrow x_1 = x_6)$ guarantees the guessed set is really an initial segment. Finally we need to say that if an element x_0 is above the initial segment $(x_1 \neq x_6)$ then there is a smaller element x_5 also above the initial segment. The mapping $x_8 \mapsto x_9$ makes sure the initial segment does not have a maximal element.

The sentence Φ_{cmpl} has many quantifier alternations but that is not really essential as we could equivalently use the universal-existential sentence:

$$
\begin{aligned}
\Phi'_{\mathrm{cmpl}} : \forall x_0 \forall x_2 \forall x_4 \forall x_8 \exists x_1 \exists x_3 & \\
\exists x_5 \exists x_6 \exists x_7 \exists x_9 (\quad & \quad {=}(x_0, x_1) \wedge {=}(x_2, x_3) \\
& \wedge \ {=}(x_4, x_5) \wedge {=}(x_6) \wedge {=}(x_7) \\
& \wedge \ {=}(x_8, x_9) \\
& \wedge \ (x_0 = x_2 \rightarrow x_1 = x_3) \\
& \wedge \ (x_1 = x_6 \rightarrow x_0 < x_6) \\
& \wedge \ (x_0 = x_7 \rightarrow x_1 = x_6) \\
& \wedge \ ((x_0 < x_2 \wedge x_3 = x_6) \rightarrow x_1 = x_6) \\
& \wedge \ ((\neg(x_1 = x_6) \wedge x_0 = x_4 \wedge x_2 = x_5) \\
& \qquad \rightarrow (x_5 < x_0 \wedge \neg(x_3 = x_6))) \\
& \wedge \ ((x_0 = x_8 \wedge x_1 = x_6 \wedge x_2 = x_9) \\
& \qquad \rightarrow (x_8 < x_9 \wedge x_3 = x_6)))
\end{aligned}
$$

Exercise 37 *Give a sentence of $\mathcal{D}$ which is true in a linear order if and only if the linear order is isomorphic to a proper initial segment of itself.*

2.4 Well-Foundedness

A binary relation R on a set M is *well-founded* if and only if there is no sequence $a_0, a_1, ...$ in M such that $a_{n+1} R a_n$ for all n, and otherwise *ill-founded*. An equivalent definition of well-foundedness is that there is no non-empty subset X of M such that for every element a in X there is an element b of X such that bRa. To express ill-foundedness we use the sentence

$$\begin{array}{rll}\Phi_{\mathrm{wf}} : \exists x_6 \exists x_7 \forall x_0 \exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (& & =\!(x_2, x_3) \\ & \wedge & =\!(x_4, x_5) \\ & \wedge & (x_0 = x_2 \rightarrow x_1 = x_3) \\ & \wedge & (x_0 = x_7 \rightarrow x_1 = x_6) \\ & \wedge & ((x_1 = x_6 \wedge x_0 = x_4 \wedge x_2 = x_5) \\ & & \rightarrow (x_3 = x_6 \wedge R x_5 x_4))\end{array}$$

The sentence Φ_{wf} is true in a binary structure (M, R) if and only if R is ill-founded.

Explanation: The mapping $x_0 \mapsto x_1$ guesses the set X as the pre-image of x_6. The mapping $x_2 \mapsto x_3$ is a copy of the mapping $x_0 \mapsto x_1$, as witnessed by $(x_0 = x_2 \rightarrow x_1 = x_3)$. x_7 manifests non-emptiness of the guessed initial segment as witnessed by $(x_0 = x_7 \rightarrow x_1 = x_6)$. The clause $((x_1 = x_6 \wedge x_0 = x_4 \wedge x_2 = x_5) \rightarrow (x_3 = x_6 \wedge R x_5 x_4))$ guarantees the guessed set has no R-smallest element,

Exercise 38 *A* partially ordered set *is an L-structure $\mathcal{M} = (M, \leq^{\mathcal{M}})$ for the vocabulary $L = \{\leq\}$, where $\leq^{\mathcal{M}}$ is assumed to be reflexive $(x \leq x)$, transitive $(x \leq y \leq z \Rightarrow x \leq z)$ and anti-symmetric $(x \leq y \leq x \Rightarrow x = y)$. We shorten $(x \leq^{\mathcal{M}} y \;\&\; x \neq y)$ to $x <^{\mathcal{M}} y$. A* chain *of a partial order is a subset of M which is linearly ordered by $\leq^{\mathcal{M}}$. Give a sentence of $\mathcal{D}$ which is true in a partially ordered set if and only if the partial order has an infinite chain.*

Exercise 39 *A* tree *is a partially ordered set $\mathcal{M}$ such that the set $\{x \in M : x <^{\mathcal{M}} t\}$ of predecessors of any $t \in M$ is well-ordered by $\leq^{\mathcal{M}}$ and there is a unique smallest element in $\mathcal{M}$, called the* root *of the tree. Thus for any $t <^{\mathcal{M}} s$ in $\mathcal{M}$ there is an immediate successor r of t such that $t <^{\mathcal{M}} r \leq^{\mathcal{M}} s$. A* subtree *of a tree is a substructure which is a tree. A tree is* binary *if every element has at most two immediate successors, and a* full binary tree *if every* *element has exactly two immediate successors. Give a sentence of $\mathcal{D}$ which is true in a tree if and only if the tree has a full binary subtree.*

Exercise 40 *The* cofinality *of a linear order is the smallest cardinal κ such that the order has an unbounded subset of cardinality κ. In particular, a linear order has cofinality ω if the linear order has a cofinal increasing sequence $a_0, a_1, \ldots$. Give a sentence of $\mathcal{D}$ which is true in a linear order if and only if the order is either ill-founded or else well-founded and of cofinality ω.*

2.5 Natural Numbers

Let P^- be the first order sentence

$$\begin{array}{l}
\forall x_0(x_0 + 0 = 0 + x_0 = x_0)\wedge \\
\forall x_0 \forall x_1(x_0 + (x_1 + 1) = (x_0 + x_1) + 1)\wedge \\
\forall x_0(x_0 \cdot 0 = 0 \cdot x_0 = 0)\wedge \\
\forall x_0 \forall x_1(x_0 \cdot (x_1 + 1) = (x_0 \cdot x_1) + x_0)\wedge \\
\forall x_0 \forall x_1(x_0 < x_1 \leftrightarrow \exists x_2(x_0 + (x_2 + 1) = x_1)) \\
\forall x_0(x_0 > 0 \rightarrow \exists x_1(x_1 + 1 = x_0))\wedge \\
0 < 1 \wedge \forall x_0(0 < x_0 \rightarrow (1 < x_0 \vee 1 = x_0))
\end{array}$$

and $\Phi_{\mathbb{N}}$ the following sentence of $\mathcal{D}$, reminiscent of Φ_∞:

$$\begin{array}{lll}
\neg P^- \vee \exists x_5 \exists x_4 \forall x_0 \exists x_1 \forall x_2 \exists x_3 (=\!(x_2, x_3) & \wedge & x_4 < x_5 \\
& \wedge & ((x_0 = x_2 \wedge x_0 < x_5) \leftrightarrow \\
& & (x_1 = x_3 \wedge x_1 < x_4)))
\end{array}$$

The sentence $\Phi_{\mathbb{N}}$ is of course true in models that do not satisfy the axiom P^-. However, in models of $\Phi_{\mathbb{N}}$ where P^- does hold, something interesting happens: the initial segments determined by x_4 and x_5 are mapped onto each other by the bijection $x_0 \mapsto x_1$. Thus such models cannot be isomorphic to $(\mathbb{N}, +, \cdot, 0, 1, <)$, in which all initial segments are finite and of different finite cardinality.

Lemma 35 *If ϕ is a sentence of dependence logic in the vocabulary of arithmetic, then the following are equivalent:*

1. *ϕ is true in $(\mathbb{N}, +, \cdot, 0, 1, <)$.*
2. *$\Phi_{\mathbb{N}} \vee \phi$ is valid in $\mathcal{D}$.*

Proof. Suppose first $\models \Phi_{\mathbb{N}} \vee \phi$. Since $(\mathbb{N}, +, \cdot, 0, 1, <) \not\models \Phi_{\mathbb{N}}$, we have necessarily $(\mathbb{N}, +, \cdot, 0, 1, <) \models \phi$. Conversely, suppose $(\mathbb{N}, +, \cdot, 0, 1, <) \models \phi$ and let $\mathcal{M}$ be arbitrary. If $\mathcal{M} \models \Phi_{\mathbb{N}}$, then trivially $\models \Phi_{\mathbb{N}} \vee \phi$. Suppose then $\mathcal{M} \not\models \Phi_{\mathbb{N}}$. Necessarily $\mathcal{M} \models P^-$. If $\mathcal{M} \not\cong (\mathbb{N}, +, \cdot, 0, 1, <)$, we get $\mathcal{M} \models \Phi_{\mathbb{N}}$ contrary to our assumption. Thus $\mathcal{M} \cong (\mathbb{N}, +, \cdot, 0, 1, <)$, whence $\mathcal{M} \models \phi$. $\Box$

If Lemma 35 is combined with Tarski's Undefinability of Truth (see Theorem 73), we obtain, using $\ulcorner \phi \urcorner$ to denote the Gödel number of ϕ according to some obvious Gödel numbering of sentences of $\mathcal{D}$:

Corollary 36 *The set $\{\ulcorner \phi \urcorner : \phi$ is valid in $\mathcal{D}\}$ is non-arithmetical.*

In particular, there cannot be any effective axiomatization of dependence logic, for then $\{\ulcorner \phi \urcorner : \phi$ is valid in $\mathcal{D}\}$ would be recursively enumerable and therefore arithmetical. We return to this important issue later.

2.6 Real Numbers

Let RF be the first order axiomatization of ordered fields. The ordered field $(\mathbb{R}, +, \cdot, 0, 1, <)$ of real numbers is the unique ordered field in which the order is a complete order. The proof of this can be found in standard textbooks on real analysis. Accordingly, let $\Phi_{\mathbb{R}}$ be the sentence $\neg RF \vee \Phi_{\mathrm{cmpl}}$ of $\mathcal{D}$. Exactly as in Lemma 35, we have:

Lemma 37 *If ϕ is a sentence of $\mathcal{D}$ in the vocabulary of ordered fields, then the following are equivalent:*

1. *ϕ is true in $(\mathbb{R}, +, \cdot, 0, 1, <)$.*

2. *$\Phi_{\mathbb{R}} \vee \phi$ is valid in $\mathcal{D}$.*

This is not as noteworthy as in the case of natural numbers, as the truth of a first order sentence in the ordered field of reals is actually effectively decidable. This is a consequence of the fact, due to Tarski, that this structure admits elimination of quantifiers (see e.g. [18]). What is noteworthy, is that we can add integers to the structure $(\mathbb{R}, +, \cdot, 0, 1, <)$, obtaining the structure $(\mathbb{R}, +, \cdot, 0, 1, <, \mathbb{N})$ with a unary predicate N for the set of natural numbers, making the first order theory of the structure undecidable, and still get a reduction as in Lemma 37. To this end, let Φ_N be

$$\begin{array}{rl} N(0) & \wedge\ \forall x_0(N(x_0) \rightarrow N(x_0+1)) \\ & \wedge\ \forall x_0(N(x_0) \rightarrow (0 = x_0 \vee 0 < x_0)) \\ & \wedge\ \forall x_0 \forall x_1((N(x_0) \wedge N(x_1) \wedge x_0 < x_1) \rightarrow \\ & \quad (x_0 + 1 = x_1 \vee x_0 + 1 < x_1)). \end{array}$$

Let $\Phi_{\mathbb{R},\mathbb{N}}$ be the sentence $\neg RF \vee \Phi_{\mathrm{cmpl}} \vee \neg \Phi_N$ of $\mathcal{D}$. Then any structure that is not a model of $\Phi_{\mathbb{R},\mathbb{N}}$ is isomorphic to $(\mathbb{R}, +, \cdot, 0, 1, <, \mathbb{N})$. Thus we obtain easily:

Lemma 38 *If ϕ is a sentence of $\mathcal{D}$ in the vocabulary of ordered fields supplemented by the unary predicate N, then the following are equivalent:*

1. *ϕ is true in $(\mathbb{R}, +, \cdot, 0, 1, <, \mathbb{N})$.*
2. *$\Phi_{\mathbb{R},\mathbb{N}} \vee \phi$ is valid in $\mathcal{D}$.*

2.7 Set Theory

The vocabulary of set theory consists of just one binary predicate symbol E. As a precursor to real set theory let us consider the following simpler situation. We have, in addition to E, also two unary predicates R and S. Let θ be the conjunction of the first order sentence $\forall x_0 \forall x_1 (x_0 E x_1 \rightarrow$

$(Rx_0 \wedge Sx_1)) \wedge \forall x_0 (Sx_0 \to \neg Rx_0)$ and the *axiom of extensionality* $\forall x_0 \forall x_1 (\forall x_2 (x_2 E x_0 \leftrightarrow x_2 E x_1) \to x_0 = x_1)$. Canonical examples of models of θ are models of the form $(M, \in, X, \mathcal{P}(X))$. Indeed, $\mathcal{M} \models \theta$ if and only if $\mathcal{M} \cong \mathcal{N}$ for some $\mathcal{N}$ such that $E^{\mathcal{N}} = \{(a,b) \in N^2 : a \in R^{\mathcal{N}}, b \in S^{\mathcal{N}}, a \in b\}$ and $S^{\mathcal{N}} \subseteq \mathcal{P}(R^{\mathcal{N}})$. Let

$$\begin{array}{rll} \Phi_{\text{ext}} : \neg\theta \vee \exists x_6 \forall x_0 \exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (& & =\!(x_2, x_3) \\ & \wedge & =\!(x_4, x_5) \\ & \wedge & (x_0 = x_2 \to x_1 = x_3) \\ & \wedge & (x_1 = x_6 \to Rx_0) \\ & \wedge & ((Sx_4 \wedge x_0 = x_5) \to \\ & & (x_5 E x_4 \nleftrightarrow x_1 = x_6))) \end{array}$$

The sentence Φ_{ext} is true in a structure $\mathcal{M}$ if and only if $\mathcal{M} \cong \mathcal{N}$ for some $\mathcal{N}$ such that $E^{\mathcal{N}} = \{(a,b) \in N^2 : a \in R^{\mathcal{N}}, b \in S^{\mathcal{N}}, a \in b\}$ and $S^{\mathcal{N}} \neq \mathcal{P}(R^{\mathcal{N}})$.

Explanation: The mapping $x_0 \mapsto x_1$ guesses a set X as the pre-image of x_6. The mapping $x_2 \mapsto x_3$ is a copy of the mapping $x_0 \mapsto x_1$, as witnessed by $(x_0 = x_2 \to x_1 = x_3)$. The clause $(x_1 = x_6 \to Rx_0)$ makes sure X is a subset of R. The clause $((Sx_4 \wedge x_0 = x_5) \to (x_5 E x_4 \nleftrightarrow x_1 = x_6))$ guarantees the guessed set is not in the set S.

Lemma 39 *If ϕ is a sentence of $\mathcal{D}$ in the vocabulary $\{E, R, S\}$, then the following are equivalent:*

1. *ϕ is true in every model of the form $(M, \in, X, \mathcal{P}(X))$.*
2. *$\Phi_{ext} \vee \phi$ is valid in $\mathcal{D}$.*

The *cumulative hierarchy* of sets is defined as follows:

$$\begin{array}{lcl} V_0 & = & \emptyset \\ V_{\alpha+1} & = & \mathcal{P}(V_\alpha) \\ V_\nu & = & \bigcup_{\beta<\alpha} V_\beta \text{ for limit } \nu \end{array} \tag{2.1}$$

Thus V_1 is the powerset of $\emptyset$, i.e. $\{\emptyset\}$, V_2 is the powerset of $\{\emptyset\}$, i.e. $\{\emptyset, \{\emptyset\}\}$, etc. The sets V_n, $n \in \mathbb{N}$, are all finite, V_ω is countable and for $\alpha > \omega$ the set V_α is uncountable. For more on the cumulative hierarchy, see Section 5.2.

Let ZFC^* be a large but finite part of the Zermelo-Fraenkel axioms for set theory (see e.g. [16] for the axioms). It follows from the axioms that every set is in some V_α. Models of the form $(V_\alpha, \in)$, α a limit ordinal, are canonical examples of models of ZFC^*.

Let

$$\begin{aligned} \Phi_{\text{set}} : \neg ZFC^* \vee \exists x_6 \exists x_7 \forall x_0 \exists x_1 \forall x_2 \exists x_3\ \forall x_4 \exists x_5 (&{=}(x_2, x_3) \wedge \\ &\wedge\ {=}(x_4, x_5) \\ &\wedge\ (x_0 = x_2 \rightarrow x_1 = x_3) \\ &\wedge\ (x_1 = x_6 \rightarrow x_0 E x_7) \\ &\wedge\ (x_0 = x_5 \rightarrow \\ &\quad (x_5 E x_4 \nleftrightarrow x_1 = x_6))) \end{aligned}$$

Lemma 40 *If ϕ is a sentence of $\mathcal{D}$ in the vocabulary $\{E\}$, then the following are equivalent:*

1. $(V_\alpha, \in) \models \phi$ *for all models* $(V_\alpha, \in)$ *of* ZFC^*.
2. $\Phi_{set} \vee \phi$ *is valid in* $\mathcal{D}$.

In consequence, there are Ψ_1, Ψ_2, Ψ_3 and Ψ_4 in dependence logic such that

1. The Continuum Hypothesis holds if and only is Ψ_1 is valid in $\mathcal{D}$
2. The Continuum Hypothesis fails if and only is Ψ_2 is valid in $\mathcal{D}$
3. There are no inaccessible cardinals if and only is Ψ_3 is valid in $\mathcal{D}$
4. There are no measurable cardinals if and only is Ψ_4 is valid in $\mathcal{D}$

These examples show that to decide whether a sentence of $\mathcal{D}$ is valid or not is tremendously difficult. On may have to search through the whole set theoretic universe. This is in sharp contrast to first order logic where to decide whether a sentence is valid or not it suffices to search through finite proofs, that is, essentially just through natural numbers.

By means of the sentence Φ_{set} it is easy to show that for any first order structure $\mathcal{M}$ definable in the set-theoretical structure $(V_{\omega\cdot 3}, \in)$, which includes virtually all commonly used mathematical structures, and any first order ϕ there is a sentence $\Phi_{\mathcal{M},\phi}$ such that the following are equivalent:

1. ϕ is true in $\mathcal{M}$.
2. $\Phi_{\mathcal{M},\phi}$ is valid in $\mathcal{D}$.

Moreover, $\Phi_{\mathcal{M},\phi}$ can be found effectively on the basis of ϕ and the defining formula of $\mathcal{M}$.

Exercise 41 *Give a sentence ϕ of $\mathcal{D}$ such that ϕ has models of all infinite cardinalities, and for all $\kappa \geq \omega$, ϕ has a unique model (up to isomorphism) of cardinality κ if and only if κ is a strong limit cardinal (i.e. $\lambda < \kappa$ implies $2^\lambda < \kappa$).*

Chapter 3

Game Theoretic Semantics

We begin with a review of the well-known game theoretic semantics of first order logic (see e.g. [9]). This is the topic of Section 3.1. There are two ways of extending the first order game to dependence logic. The first, presented in Section 3.2, corresponds to the transition in semantics from assignments to teams. The second game theoretic semantics for dependence logic is closer to the original semantics of independence friendly logic presented in [8, 10]. In the second game theoretic formulation the dependence relation $=(x_0, ..., x_n)$ does not come up as an atomic formula but as the possibility to incorporate *imperfect information* into the game. A player who aims at securing $=(x_0, ..., x_n)$ when the game ends has to be able to choose a value for x_n only on the basis of what the values of $x_0, ..., x_{n-1}$ are. In this sense the player's *information set* is restricted to $x_0, ..., x_{n-1}$ when he or she chooses x_n.

3.1 The Semantic Game of First Order Logic

The game theoretic semantics of first order logic has a long history. The basic idea is that if a sentence is true, its truth, asserted by us, can be defended against a doubter. A doubter can question the truth of a conjunction $\phi \wedge \psi$ by doubting the truth of, say, ψ. He can doubt the truth of a disjunction $\phi \vee \psi$ by asking which of ϕ and ψ is the one that is true. He can doubt the truth of a negation $\neg\phi$ by claiming that ϕ is true instead of $\neg\phi$. At this point we become the doubter and start questioning why ϕ is true. This interaction can be formulated in terms of a simple game between two players that we call players **I** and **II**. We call them opponents of each other. The opponent of player α is denoted by α^*. In the literature these are sometimes called Abelard and Eloise. We go along to the extent that we refer to player **I** as “he” and to player **II** as “she.” The players observe a formula ϕ and an assignment s in the context of a given model $\mathcal{M}$. In the beginning of the game player **II** claims that assignment s satisfies ϕ in $\mathcal{M}$, and player **I** doubts this. During the game their roles may change, as we just saw in the case of negation. To keep track of who is claiming what we use the notation (ϕ, s, α) for a position in the game. Here α is either **I** or **II**. The idea is that α indicates which player is claiming that s satisfies ϕ in $\mathcal{M}$.

Definition 41 *The semantic game $H(\phi)$ of first order logic in a model $\mathcal{M}$ is the following game: There are two players, **I** and **II**. A position of the game is a triple (ψ, s, α), where ψ is a subformula of $\emptyset$, s is an assignment the domain of which contains the free variables of ψ, and $\alpha \in \{\mathbf{I}, \mathbf{II}\}$. In the beginning of the game the position is $(\phi, \emptyset, \mathbf{II})$. The rules of the game are as follows:*

1. *The position is $(t_1 = t_2, s, \alpha)$: If $t_1^{\mathcal{M}}\langle s\rangle = t_2^{\mathcal{M}}\langle s\rangle$, then player α wins and otherwise the opponent wins.*
2. *The position is $(R(t_1, \ldots, t_n), s, \alpha)$: If s satisfies $R(t_1, \ldots, t_n)$ in $\mathcal{M}$, then player α wins, otherwise the opponent wins.*
3. *The position is $(\neg\psi, s, \alpha)$: The game switches to the position (ψ, s, α^*), where α^* is the opponent of α.*
4. *The position is $(\psi \vee \theta, s, \alpha)$: The next position is (ψ, s, α) or (θ, s, α), and α decides which.*
5. *The position is $(\exists x_n \psi, s, \alpha)$: Player α chooses $a \in M$ and the next position is $(\psi, s(a/x_n), \alpha)$.*

The above game is a *zero-sum* game, i.e. one player's loss is the other player's victory. It is also a game of *perfect information* in the sense the strategies of both players are allowed to depend on the whole sequence of previous positions. By the *Gale-Stewart Theorem* [5] all finite *zero-sum games of perfect information are determined.* All the possible positions of this game form in a canonical way a tree, which we call the *game tree*. The game tree for $H(\phi)$ starts from the position $(\phi, \emptyset, \mathbf{II})$. Any (maximal) branch of this tree represents a possible *play* of the game. We call proper initial segments of plays *partial* plays.

Inspection of the game tree is vital for success in a game. It is clear that in order to be able to declare victory a player has to have a clear picture in his or her mind what to play in each position. The following concept of strategy is the heart of game theory. It is a mathematically exact concept which tries to capture the idea of a player knowing what to play in each position.

Definition 42 *A* strategy *of player α in $H(\phi)$ is any sequence τ of functions τ_i defined on the set of all partial plays $(p_0, ..., p_{i-1})$ satisfying:*

- *If $p_{i-1} = (\phi \vee \psi, s, \alpha)$, then τ tells player α which formula to pick, i.e. $\tau_i(p_0, ..., p_{i-1}) \in \{0, 1\}$. If the strategy gives value 0, player α picks the left-hand*[1] *formula ϕ, and otherwise the right-hand formula ψ.*
- *If $p_{i-1} = (\exists x_n \phi, s, \alpha)$, then τ tells player α which element $a \in M$ to pick, that is $\tau_i(p_0, ..., p_{i-1}) \in M$.*

We say that player α has used *strategy τ in a play of the game $H(\phi)$ if in each relevant case player α has used τ to make his or her choice. More exactly, player α has used τ in a play $p_0, ..., p_n$ if the following two conditions hold for all $i < n$:*

[1] One may ask why the values of the strategy are numbers 0 and 1 rather than the formulas themselves. The reason is that the formulas may be one and the same. It is a delicate point whether it then makes any difference which formula is picked. For first order logic there is no difference but there are extensions of first order logic where a difference emerges.

- *If* $p_{i-1} = (\phi \vee \psi, s, \alpha)$ *and* $\tau_i(p_0, ..., p_{i-1}) = 0$*, then* $p_i = (\phi, s, \alpha)$*, while if* $\tau_i(p_0, ..., p_{i-1}) = 1$*, then* $p_i = (\psi, s, \alpha)$*.*
- *If* $p_{i-1} = (\exists x_m \phi, s, \alpha)$ *and* $\tau_i(p_0, ..., p_{i-1}) = a$*, then* $p_i = (\phi, s(a/x_m), \alpha)$*.*

A strategy τ *of player* α *in the game* $H(\phi)$ *is a* winning strategy*, if player* α *wins every play in which he or she has used* τ*.*

Note that the property of a strategy τ being a winning strategy is defined without any reference to actual playing of the game. This is not an oversight but an essential feature of the mathematical theory of games. We have reduced the intuitive act of players choosing their moves to combinatorial properties of some functions. One has to be rather careful in such a reduction. It is possible that the act of playing and handling formulas may use some property of formulas that is intuitively evident but not coded by the mathematical model. One such potential property is the place of a subformula in a formula. We will return to this point later.

Theorem 43 *Suppose* ϕ *is a sentence of first order logic. Then* $\mathcal{M} \models_\emptyset \phi$ *in first order logic if and only if player* **II** *has a winning strategy in the semantic game* $H(\phi)$*.*

Proof. Suppose $\mathcal{M} \models_\emptyset \phi$ in first order logic. Consider the following strategy of player **II**. She maintains the condition:

($\star$) If the position is $(\phi, s, \mathbf{II})$, then $\mathcal{M} \models_s \phi$. If the position is $(\phi, s, \mathbf{I})$, then $\mathcal{M} \models_s \neg\phi$.

It is completely routine to check that **II** can actually follow this strategy and win. Note that in the beginning $\mathcal{M} \models_\emptyset \phi$, so $(\star)$ holds.

For the other direction, suppose player **II** has a winning strategy τ in the semantic game starting from $(\phi, \emptyset, \mathbf{II})$. It is again completely routine to show:

(⋆⋆) If a position (ψ, s, α) is reached in the game, player **II** using τ, then $\mathcal{M} \models_s \psi$ if $\alpha = \mathbf{II}$, and $\mathcal{M} \models_s \neg\psi$ if $\alpha = I$.

Since the initial position $(\phi, \emptyset, \mathbf{II})$ is reached in the beginning of the game, we obtain from (⋆⋆) the desired conclusion $\mathcal{M} \models_\emptyset \phi$. □

Exercise 42 *Consider the game $H(\exists x_0 \forall x_1 (x_0 = x_1 \vee x_0 E x_1))$ in the graph below. Draw the game tree and use it to describe the winning strategy of the player who has it.*

$$\overline{1 \quad 2 \quad} 3$$

Exercise 43 *Let L consist of two unary predicates P and R. Let $\mathcal{M}$ be an L-structure such that $M = \{0,1,2,3\}$, $P^{\mathcal{M}} = \{0,1,2\}$ and $R^{\mathcal{M}} = \{1,2,3\}$. Who has a winning strategy in $H(\phi)$ if ϕ is $\exists x_0 (Px_0 \wedge Rx_0)$?, $\forall x_0 \exists x_1 \neg(x_0 = x_1)$? Describe the winning strategy.*

Exercise 44 *Suppose $\mathcal{M}$ is the binary structure $(\{0,1,2\}, R)$, where R is as in Figure 3.1. Who has a winning strategy in $G(\phi)$ if ϕ is $\forall x_0 \exists x_1 (\neg R x_0 x_1 \wedge \forall x_2 \exists x_3 (\neg R x_2 x_3))$? Describe the winning strategy.*

Figure 3.1: A binary structure

Exercise 45 *Suppose $\mathcal{M}$ is the binary structure $(\{0,1,2\}, R)$, where R is as in Figure 3.1. Who has a winning strategy in $H(\phi)$ if ϕ is $\exists x_0 \forall x_1 (R x_0 x_1 \vee \exists x_2 \forall x_3 (R x_2 x_3))$. Describe the winning strategy.*

Exercise 46 *Show that if τ is a strategy of player* **II** *in $H(\phi)$ and σ is a strategy of player* **I** *in $H(\phi)$, then there is one and only one play of $H(\phi)$ in which player* **II** *has used τ and player* **I** *has used σ. We denote this play by $[\tau, \sigma]$.*

Exercise 47 *Show that a strategy τ of player* **II** *in $G(\phi)$ is a winning strategy if and only if player* **II** *wins the play $[\tau, \sigma]$ for every strategy σ of player I.*

3.2 A Perfect Information Game for Dependence Logic

In this section we define a game of perfect information, introduced in [27]. This game is very close to our definition of the semantics of dependence logic. In this game the moves are triples (ϕ, X, d), where X is a team, ϕ is a formula and $d \in \{0, 1\}$. If ϕ is a conjunction and $d = 0$, we may have an ordered pair of teams. This game has less symmetry than $H(\phi)$. On the other hand, the game has formulas of dependence logic as arguments, and $\mathcal{D}$ does not enjoy the same kind of symmetry as first order logic. In particular, we cannot let negation correspond to exchanging the roles of the players, as in the case of $H(\phi)$, and at the same time have sentences which are neither true nor false.

The game we are going to define has two players called **I** and **II**. A *position* in the game is a triple $p = (\phi, X, d)$, where ϕ is a formula, X is a team, the free variables of ϕ are in $\mathrm{dom}(X)$, and $d \in \{0, 1\}$.

Definition 44 *Let $\mathcal{M}$ be a structure. The game $G(\phi)$ is defined by the following inductive definition for all sentences ϕ of dependence logic. The type of the move of each player is determined by the position as follows:*

(M1) *The position is $(\phi, X, 1)$ and $\phi = \phi(x_{i_1}, ..., x_{i_k})$ is of the form $t_1 = t_2$ or of the form $Rt_1...t_n$. Then the game ends. Player* **II** *wins if*

$$(\forall s \in X)(\mathcal{M} \models \phi(s(x_{i_1}), ..., s(x_{i_k}))).$$

Otherwise player **I** *wins.*

(M2) *The position is $(\phi, X, 0)$ and $\phi = \phi(x_{i_1}, ..., x_{i_k})$ is of the form $t_1 = t_2$ or of the form $Rt_1...t_n$. Then the game ends. Player* **II** *wins if*

$$(\forall s \in X)(\mathcal{M} \not\models \phi(s(x_{i_1}), ..., s(x_{i_k}))).$$

Otherwise player **I** *wins.*

(M3) *The position is* $(=(t_1, ..., t_n), X, 1)$*. Then the game ends. Player* **II** *wins if* $\mathcal{M} \models_X =(t_1, ..., t_n)$*. Otherwise player* **I** *wins.*

(M4) *The position is* $(=(t_1, ..., t_n), X, 0)$*. Then the game ends. Player* **II** *wins if* $X = \emptyset$*. Otherwise player* **I** *wins.*

(M5) *The position is* $(\neg\phi, X, 1)$*. The game continues from the position* $(\phi, X, 0)$*.*

(M6) *The position is* $(\neg\phi, X, 0)$*. The game continues from the position* $(\phi, X, 1)$*.*

(M7) *The position is* $(\phi \vee \psi, X, 1)$*. Now player* **II** *chooses* Y *and* Z *such that* $X = Y \cup Z$*, and we move to position* $(\phi \vee \psi, (Y, Z), 1)$*. Then player* **I** *chooses whether the game continues from position* $(\phi, Y, 1)$ *or* $(\psi, Z, 1)$*.*

(M8) *The position is* $(\phi \vee \psi, X, 0)$*. Now player* **I** *chooses whether the game continues from position* $(\phi, X, 0)$ *or* $(\psi, X, 0)$*.*

(M9) *The position is* $(\exists x_n \phi, X, 1)$*. Now player* **II** *chooses* $F : X \to M$ *and then the game continues from the position* $(\phi, X(F/x_n), 1)$*.*

(M10) *The position is* $(\exists x_n \phi, X, 0)$*. Now the game continues from the position* $(\phi, X(M/x_n), 0)$*.*

In the beginning of the game, the position is $(\phi, \{\emptyset\}, 1)$*.*

Note that case (M8) generates two rounds for the game: during the first round player **II** makes a choice for Y and Z. We call this round a *half-round*. During the next round player **I** makes a choice between them. Thus after the position $(\phi \vee \psi, X, 1)$ there is the position $(\phi \vee \psi, (Y, Z), 1)$, from which the game then proceeds to either $(\phi, Y, 1)$ or $(\psi, Z, 1)$. Note also that player **II** has something to do only in the cases (M7) and (M9). Likewise, player **I** has something to do only in cases (M7) and (M8). Otherwise the game goes on in a determined way with no interaction from the players.

All the possible positions of this game form a tree, just as in the case of the game $H(\phi)$. The tree for $G(\phi)$ starts from the position $(\phi, \{\emptyset\}, 1)$. Depending on ϕ it continues in different way. Note that player **II** has always a winning strategy in a position of the form $(\phi, \emptyset, d)$.

Example 45 *Suppose $\mathcal{M}$ has at least 2 elements. Player* **I** *has a winning strategy in $G(\forall x_0 \exists x_1 (=(x_1) \wedge x_0 = x_1))$. Note that when (M9) is applied, the tree splits into as many branches as there are functions F. In the end of the game the winner is decided on the basis of (M1)-(M4).*

We define now what we mean by a strategy in the game $G(\phi)$.

Definition 46 *A* strategy *of player* **II** *in $G(\phi)$ is any sequence τ of functions τ defined on the set of all partial plays $(p_0, ..., p_{i-1})$ satisfying:*

- *If $p_{i-1} = (\phi \vee \psi, X, 1)$, then τ tells player* **II** *how to cover X with two sets, one corresponding to ϕ and the other to ψ, i.e. $\tau_i(p_0, ..., p_{i-1}) = (Y, Z)$ such that $X = Y \cup Z$.*
- *If $p_{i-1} = (\exists x_n \phi, X, 1)$, then τ tells player* **II** *how to supplement X, i.e. $\tau_i(p_0, ..., p_{i-1})$ is a function $F : X \to M$.*

We say that player **II** *has used strategy τ in a play of the game $G(\phi)$ if in the cases (M7) and (M9) player* **II** *has used τ to make the choice. More exactly, player* **II** *has used τ in a play $p_0, ..., p_n$ if the following two conditions hold for all $i < n$:*

- *If $p_{i-1} = (\phi \vee \psi, X, 1)$ and $\tau_i(p_0, ..., p_{i-1}) = (Y, Z)$, then $p_i = (\phi \vee \psi, (Y, Z), 1)$.*
- *If $p_{i-1} = (\exists x_n \phi, X, 1)$ and $\tau_i(p_0, ..., p_{i-1}) = F$, then $p_i = (\phi, X(F/x_n), 1)$.*

A strategy *of player* **I** *in $G(\phi)$ is any sequence σ of functions σ_i defined on the set of all partial plays $p_0, ..., p_{i-1}$ satisfying:*

- *If $p_{i-1} = (\phi \vee \psi, X, 0)$, then σ tells player* **I** *which formula to pick, i.e.* $\sigma_i(p_0, ..., p_{i-1}) \in \{0, 1\}$. *If the strategy gives value* 0, *player* **I** *picks the left-hand formula ϕ, and otherwise the right-hand formula ψ.*
- *If $p_{i-1} = (\phi \vee \psi, (Y, Z), 1)$, then $\sigma_i(p_0, ..., p_{i-1}) \in \{0.1\}$.*

We say that player **I** *has used strategy σ in a play of the game $G(\phi)$ if in the cases (M7) and (M8) player* **I** *has used σ to make the choice. More exactly,* **I** *has used σ in a play $p_0, ..., p_n$ if the following two conditions hold:*

- *If $p_{i-1} = (\phi \vee \psi, X, 0)$ and $\sigma_i(p_0, ..., p_{i-1}) = 0$, then $p_i = (\phi, X, 0)$, and if $\sigma_i(p_0, ..., p_{i-1}) = 1$, then $p_i = (\psi, X, 0)$.*
- *If $p_{i-1} = (\phi \vee \psi, (Y, Z), 1)$, then $p_i = (\phi, Y, 0)$, if $\sigma_i(p_0, ..., p_{i-1}) = 0$, and $p_i = (\psi, Z, 0)$, if $\sigma_i(p_0, ..., p_{i-1}) = 1$.*

A strategy of player α in the game $G(\phi)$ is a winning strategy, *if player α wins every play in which she has used the strategy.*

Theorem 47 *$\mathcal{M} \models \phi$ if and only if player* **II** *has a winning strategy in $G(\phi)$.*

Proof. Assume first $\mathcal{M} \models \phi$. We describe a winning strategy of player **II** in $G(\phi)$. Player **II** maintains in $G(\phi)$ the condition that if the position (omitting half-rounds) is (ψ, X, d), then $(\psi, X, d) \in \mathcal{T}$. We prove this by induction on ϕ:

S1 Position $(\phi, X, 1)$, where ϕ is $t = t'$ or $Rt_1...t_n$. Since $(\phi, X, 1) \in \mathcal{T}$, player **II** wins, by Definition 44 (M1).

S2 Position $(\phi, X, 0)$, where ϕ is $t = t'$ or $Rt_1...t_n$. Since $(\phi, X, 0) \in \mathcal{T}$, player **II** wins, by Definition 44 (M2).

S3 Position $(=(t_1, ..., t_n), X, 1)$. Since $(=(t_1, ..., t_n), X, 1) \in \mathcal{T}$, player **II** wins, by definition.

S4 Position $(=(t_1, ..., t_n), X, 0)$. Since $(=(t_1, ..., t_n), X, 0) \in \mathcal{T}$, we have $X = \emptyset$ and therefore **II** wins by definition.

S5 Position $(\neg\phi, X, 1)$. Since $(\neg\phi, X, 1) \in \mathcal{T}$, we have $(\phi, X, 0) \in \mathcal{T}$. Thus **II** can play this move according to her plan.

S6 Position $(\neg\phi, X, 0)$. Since $(\neg\phi, X, 0) \in \mathcal{T}$, we have $(\phi, X, 1) \in \mathcal{T}$. Thus **II** can play this move according to her plan.

S7 Position $(\phi \vee \psi, X, 0)$. We know $(\phi \vee \psi, X, 0) \in \mathcal{T}$ and therefore both $(\phi, X, 0) \in \mathcal{T}$ and $(\psi, X, 0) \in \mathcal{T}$. Thus whether the game proceeds to $(\phi, X, 0)$ or $(\psi, X, 0)$, player **II** maintains her plan.

S8 Position $(\phi \vee \psi, X, 1)$. We know $(\phi \vee \psi, X, 1) \in \mathcal{T}$, and hence $(\phi, Y, 1) \in \mathcal{T}$ and $(\psi, Z, 1) \in \mathcal{T}$ for some Y and Z with $X = Y \cup Z$. Thus we can let player **II** play the ordered pair (Y, Z). After this half-round player **I** wants the game to proceed either to $(\phi, Y, 1)$ or to $(\psi, Z, 1)$. In either case player **II** can fulfil her plan, as both $(\phi, Y, 1) \in \mathcal{T}$ and $(\psi, Z, 1) \in \mathcal{T}$.

S9 Position $(\exists x_n\phi, X, 1)$. Thus there is $F : X \to M$ such that the triple $(\phi, X(F/x_n), 1)$ is in $\mathcal{T}$. Player **II** can now play the function F, for in the resulting position $(\phi, X(F/x_n), 1)$ she can maintain the condition $(\phi, X(F/x_n), 1) \in \mathcal{T}$.

S10 Position $(\exists x_n\phi, X, 0)$. We know $(\phi, X(M/x_n), 0) \in \mathcal{T}$. But the triple $(\phi, X(M/x_n), 0)$ is the next position, so **II** can maintain her plan.

For the other direction, we assume player **II** has a winning strategy τ in $G(\phi)$ and use this to show $\mathcal{M} \models \phi$. We prove by induction on ϕ that if **II** is using τ and a position (ψ, X, d) is reached, then $(\psi, X, d) \in \mathcal{T}$. This gives the desired conclusion as the initial position $(\phi, \{\emptyset\}, 1)$ is trivially reached.

S1′ Position $(\phi, X, 1)$, where ϕ is $t = t'$ or $Rt_1...t_n$. Since **II** has been playing her winning strategy, $(\phi, X, 1) \in \mathcal{T}$ by Definition 44 (M1).

S2′ Position $(\phi, X, 0)$, where ϕ is $t = t'$ or $Rt_1...t_n$. Since **II** has been playing her winning strategy, $(\phi, X, 0) \in \mathcal{T}$ by Definition 44 (M2).

S3′ Position $(=(t_1, ..., t_n), X, 1)$. Since **II** has been playing her winning strategy, $(=(t_1, ..., t_n), X, 1) \in \mathcal{T}$ by definition.

S4′ Position $(=(t_1, ..., t_n), X, 0)$. Since **II** is winning, $X = \emptyset$, and therefore $(=(t_1, ..., t_n), X, 0) \in \mathcal{T}$ by definition.

S5′ Position $(\neg\phi, X, 1)$. The game continues, still following τ, to the position $(\phi, X, 0)$. By the induction hypothesis, $(\phi, X, 0) \in \mathcal{T}$, and therefore $(\neg\phi, X, 1) \in \mathcal{T}$.

S6′ Position $(\neg\phi, X, 0)$. The game continues, still following τ, to the position $(\phi, X, 1)$. By the induction hypothesis, $(\phi, X, 1) \in \mathcal{T}$, and therefore $(\neg\phi, X, 0) \in \mathcal{T}$.

S7′ Position $(\phi \vee \psi, X, 0)$. To prove $(\phi \vee \psi, X, 0) \in \mathcal{T}$, we need both $(\phi, X, 0) \in \mathcal{T}$ and $(\psi, X, 0) \in \mathcal{T}$. Let us try to prove $(\phi, X, 0) \in \mathcal{T}$. Since **II** is following a winning strategy, we can let the game proceed to position $(\phi, X, 0)$. By the induction hypothesis, $(\phi, X, 0) \in \mathcal{T}$. The same argument gives $(\psi, X, 0) \in \mathcal{T}$. Thus we have proved $(\phi \vee \psi, X, 0) \in \mathcal{T}$.

S8′ Position $(\phi \vee \psi, X, 1)$. To get $(\phi \vee \psi, X, 1) \in \mathcal{T}$ we need $(\phi, Y, 1) \in \mathcal{T}$ and $(\psi, Z, 1) \in \mathcal{T}$ for some Y and Z with $X = Y \cup Z$. Indeed, the winning strategy τ gives an ordered pair (Y, Z) with $X = Y \cup Z$. Let us try to prove $(\phi, Y, 1) \in \mathcal{T}$. Since **II** is following a winning strategy, we can let the game proceed to position $(\phi, Y, 1)$. By the induction hypothesis, $(\phi, Y, 1) \in \mathcal{T}$. The same argument gives $(\psi, Z, 1) \in \mathcal{T}$. Thus we have proved $(\phi \vee \psi, X, 1) \in \mathcal{T}$.

S9′ Position $(\exists x_n\phi, X, 1)$. The strategy τ gives $F : X \to M$ such that **II** has a winning strategy in position $(\phi, X(F/x_n), 1)$. By the induction hypothesis $(\phi, X(F/x_n), 1) \in \mathcal{T}$. Hence $(\exists x_n\phi, X, 1) \in \mathcal{T}$.

S10′ Position $(\exists x_n\phi, X, 0)$. The game continues, still following τ, to the position $(\phi, X(M/x_n), 0)$. By the induction hypothesis, the triple $(\phi, X(M/x_n), 0)$ is in $\mathcal{T}$, and therefore $(\exists x_n\phi, X, 0) \in \mathcal{T}$.

$\square$

Theo Janssen [15] has pointed out the following example:

Example 48 *The sentence*

$$\forall x_0 \exists x_1((=(x_1) \wedge \neg x_0 = x_1) \vee (=(x_1) \wedge \neg x_0 = x_1)) \tag{3.1}$$

is true in the natural numbers. The trick is the following: For $s \in \{\emptyset\}(\mathbb{N}/x_0)$ *let* $F(s) \in \{0,1\}$ *be such that* $F(s) \neq s(x_0)$. *The team* $\{\emptyset\}(\mathbb{N}/x_0)(F/x_1)$ *is a subset of the union of* $\{\emptyset\}(\mathbb{N}/x_0)(F_0/x_1)$ *and* $\{\emptyset\}(\mathbb{N}/x_0)(F_1/x_1)$, *where* F_0 *is the constant function* 0 *and* F_1 *is the constant function* 1. *Both parts satisfy* $(=(x_1) \wedge \neg x_0 = x_1)$.

The above example shows that when we play a game that follows the structure of a formula, we may have to take the formula structure into the game. To accomplish this and in order to be sufficiently precise, we identify formulas with finite strings of symbols. Variables x_n are treated as separate symbols. The other symbols are the symbols of the vocabulary, $=,), (, \neg, \wedge, \exists$ and the comma. Each string S has a length, which we denote by $\mathrm{len}(S)$. We number the symbols in a formula with positive integers starting from the left, as in:

$(=(x_0) \vee \neg x_0 = x_1)$										
(	=	(	x_0	)	$\vee$	$\neg$	x_0	=	x_1	)
1	2	3	4	5	6	7	8	9	10	11

If the n[th] symbol of ϕ starts a string which is a subformula of ϕ, we denote the subformula by $\Lambda(\phi, n)$. Thus every subformula of ϕ is of the form $\Lambda(\phi, n)$ for some n and some may occur with several n. In the case that ϕ is the formula (3.1), the subformula $(=(x_1) \wedge \neg x_0 = x_1)$ occurs as both $\Lambda(\phi, 6)$ and $\Lambda(\phi, 18)$. Note that:

1. If $\Lambda(\phi, n) = \neg\psi$, then $\Lambda(\phi, n+1) = \psi$
2. If $\Lambda(\phi, n) = (\psi \vee \theta)$, then $\Lambda(\phi, n+1) = \psi$ and $\Lambda(\phi, n+2+\text{len}(\psi)) = \theta$.
3. If $\Lambda(\phi, n) = \exists x_m \psi$, then $\Lambda(\phi, n+2) = \psi$.

We let $G_{\text{place}}(\phi)$ be the elaboration of the game $G(\phi)$ in which the rules are the same as in $G(\phi)$ but the positions are of the form (ψ, n, X, d), and it is assumed all the time that $\Lambda(\phi, n) = \psi$. Thus the formula ψ could be computed from the number n and it is mentioned in the position only for the sake of clarity.

Definition 49 *Let $\mathcal{M}$ be a structure. The game $G_{\textit{place}}(\phi)$ is defined by the following inductive definition for all sentences ϕ of dependence logic. The type of the move of each player is determined by the position as follows:*

(M1′) *The position is $(\phi, n, X, 1)$ and $\phi = \phi(x_{i_1}, ..., x_{i_k})$ is of the form $t_1 = t_2$ or of the form $Rt_1...t_n$. Then the game ends. Player* **II** *wins if*

$$(\forall s \in X)(\mathcal{M} \models \phi(s(x_{i_1}), ..., s(x_{i_k}))).$$

Otherwise player **I** *wins.*

(M2′) *The position is $(\phi, n, X, 0)$ and $\phi = \phi(x_{i_1}, ..., x_{i_k})$ is of the form $t_1 = t_2$ or of the form $Rt_1...t_n$. Then the game ends. Player* **II** *wins if*

$$(\forall s \in X)(\mathcal{M} \not\models \phi(s(x_{i_1}), ..., s(x_{i_k}))).$$

Otherwise player **I** *wins.*

(M3′) *The position is $({=}(t_1, ..., t_n), m, X, 1)$. Then the game ends. Player* **II** *wins if $\mathcal{M} \models_X {=}(t_1, ..., t_n)$. Otherwise player* **I** *wins.*

(M4′) *The position is $({=}(t_1, ..., t_n), m, X, 0)$. Then the game ends. Player* **II** *wins if $X = \emptyset$. Otherwise player* **I** *wins.*

(M5′) *The position is* $(\neg\phi, n, X, 1)$. *The game continues from the position* $(\phi, n+1, X, 0)$.

(M6′) *The position is* $(\neg\phi, n, X, 0)$. *The game continues from the position* $(\phi, n+1, X, 1)$.

(M7′) *The position is* $(\phi \vee \psi, n, X, 0)$. *Now player* **I** *chooses whether the game continues from position* $(\phi, n+1, X, 0)$ *or* $(\psi, n+2+\mathrm{len}(\phi), X, 0)$.

(M8′) *The position is* $(\phi \vee \psi, n, X, 1)$. *Now player* **II** *chooses* X_0 *and* X_1 *such that* $X = X_0 \cup X_1$, *and we move to position* $(\phi \vee \psi, n, (X_0, X_1), 1)$. *Then player* **I** *chooses whether the game continues from position* $(\phi, n+1, X_0, 1)$ *or* $(\psi, n+2+\mathrm{len}(\phi), X_1, 1)$.

(M9′) *The position is* $(\exists x_n \phi, m, X, 1)$. *Now player* **II** *chooses* $F : X \to M$ *and then the game continues from the position* $(\phi, m+2, X(F/x_n), 1)$.

(M10′) *The position is* $(\exists x_n \phi, m, X, 0)$. *Now the game continues from the position* $(\phi, m+2, X(M/x_n), 0)$.

In the beginning of the game, the position is $(\phi, 1, \{\emptyset\}, 1)$.

The following easy observation shows that coding the location of the subformula into the game makes no difference. However, we shall use $G_{\text{place}}(\phi)$ later.

Proposition 50 *If player* **II** *has a winning strategy in* $G(\phi)$, *she has a winning strategy in* $G_{\text{place}}(\phi)$, *and vice versa.*

Proof. Assume **II** has winning strategy τ in $G(\phi)$. We describe her winning strategy in $G_{\text{place}}(\phi)$. While playing $G_{\text{place}}(\phi)$ she also plays $G(\phi)$ maintaining the condition that if the position in $G_{\text{place}}(\phi)$ is (ϕ, n, X, d), then the position in $G(\phi)$ is (ϕ, X, d) while she uses τ in $G(\phi)$.

1. Position is (ϕ, n, X, d), where ϕ is $t = t'$ or $Rt_1 ... t_n$. Player **II** wins since by assumption she wins $G(\phi)$ in position (ϕ, X, d).

2. Position is $(=(t_1, ..., t_n), n, X, d)$. Player **II** wins as we assume she wins $G(\phi)$ in position $(=(t_1, ..., t_n), X, d)$.

3. Position is $(\neg\phi, n, X, d)$. Player **II** moves to the position $(\phi, n+1, X, 1-d)$ in $G_{\text{place}}(\phi)$ and to the position $(\phi, X, 1-d)$ in $G(\phi)$. Her strategy is still valid.

4. Position is $((\phi \vee \psi), n, X, 0)$. Player **II** proceeds, according to the choice of player **I**, to the position $(\phi, n+1, X, 0)$ or to the position $(\psi, n+2+\text{len}(\phi), X, 0)$ in $G_{\text{place}}(\phi)$ and, respectively to the position $(\phi, X, 0)$ or to the position $(\psi, X, 0)$ in $G(\phi)$. Whichever way the game proceeds, her strategy is still valid.

5. Position is $((\phi \vee \psi), n, X, 1)$. We know that **II** is using τ in $G(\phi)$ so in the position $((\phi \vee \psi), X, 1) \in \mathcal{T}$ she has Y and Z with $X = Y \cup Z$, she can still win with τ from positions $(\phi, Y, 1)$ and $(\psi, Z, 1)$. Thus we let player **II** play the ordered pair (Y, Z). After this half-round player **I** wants the game to proceed either to $(\phi, n+1, Y, 1)$ or to $(\psi, n+2+\text{len}(\phi), Z, 1)$. In either case player **II** can maintain her plan.

6. Position is $(\exists x_n \phi, m, X, 1)$. The strategy τ gives player **II** a function $F : X \to M$ and the game $G(\phi)$ proceeds to the position $(\phi, X(F/x_n), 1)$. We let **II** play this function F. The game $G_{\text{place}}(\phi)$ proceeds to the position $(\phi, m+2, X(F/x_n), 1)$ and **II** maintains her plan.

7. Position is $(\exists x_n \phi, m, X, 0)$. Player **II** proceeds to the position $(\phi, m+2, X(M/x_n), 0)$ in $G_{\text{place}}(\phi)$ and to the respective position $(\phi, X(M/x_n), d)$ in $G(\phi)$. Player **II** maintains her plan.

For the other direction, assume **II** has winning strategy τ in $G_{\text{place}}(\phi)$. We describe her winning strategy in $G(\phi)$. While playing $G(\phi)$ she also plays $G_{\text{place}}(\phi)$ maintaining the condition that if the position in $G(\phi)$ is

(ϕ, X, d), then the position in $G(\phi)$ is (ϕ, n, X, d) for some n while she uses τ in $G_{\text{place}}(\phi)$.

1. Position is (ψ, X, d), where ψ is atomic. Player **II** wins since by assumption she wins $G_{\text{place}}(\phi)$ in position (ψ, n, X, d) for some n.
2. Position in $G_{\text{place}}(\phi)$ is $(\neg\phi, X, d)$ and in $G_{\text{place}}(\phi)$ it is $(\neg\phi, n, X, d)$. Player **II** moves to the position $(\phi, n+1, X, 1-d)$ in $G_{\text{place}}(\phi)$ and to the position $(\phi, X, 1-d)$ in $G(\phi)$. Her strategy is still valid.
3. Position is $((\phi \vee \psi), X, 0)$ and in $G_{\text{place}}(\phi)$ it is $((\phi \vee \psi), n, X, 0)$. Player **II** moves to the position $(\phi, X, 0)$ or to the position $(\psi, X, 0)$ in $G_{\text{place}}(\phi)$ and, respectively to the position $(\phi, n+1, X, 0)$ or to the position $(\psi, n+2+\text{len}(\phi), X, 0)$ in $G(\phi)$. Whichever way the game proceeds, her strategy is still valid.
4. Position is $((\phi \vee \psi), X, 1)$ and in $G_{\text{place}}(\phi)$ it is $((\phi \wedge \psi), n, X, 1)$. We know that **II** is using τ in $G_{\text{place}}(\phi)$ so in the position $((\phi \vee \psi), n, X, 1) \in \mathcal{T}$ she has Y and Z with $X = Y \cup Z$, she can still win with τ from positions $(\phi, n+1, Y, 1)$ and $(\psi, n+2+\text{len}(\phi), Z, 1)$. Thus we let player **II** play the ordered pair (Y, Z). After this half-round player **I** wants the game to proceed either to $(\phi, Y, 1)$ or to $(\psi, Z, 1)$. Player **II** moves in $G_{\text{place}}(\phi)$ respectively to $(\phi, n+1, Y, 1)$ or to $(\psi, n+2+\text{len}(\phi), Z, 1)$. In either case player **II** can maintain her plan.
5. Position is $(\exists x_n\phi, X, 1)$ and in $G_{\text{place}}(\phi)$ it is $(\exists x_n\phi, m, X, 1)$. The strategy τ gives player **II** a function $F : X \to M$ and the game $G_{\text{place}}(\phi)$ proceeds to the position $(\phi, m+2, X(F/x_n), 1)$. We let **II** play this function F. The game $G(\phi)$ proceeds to the position $(\phi, X(F/x_n), 1)$ and **II** maintains her plan.
6. Position is $(\exists x_n\phi, X, 0)$ and in $G_{\text{place}}(\phi)$ it is $(\exists x_n\phi, m, X, 0)$. Player **II** proceeds to the position $(\phi, m+2, X(M/x_n), 0)$ in $G_{\text{place}}(\phi)$ and

to the position $(\phi, X(M/x_n), d)$ in $G(\phi)$. Player **II** maintains her plan.

$\square$

Lemma 51 *Suppose player* **II** *uses the strategy* τ *in* $G_{\text{place}}(\phi)$ *and the game reaches a position* (ψ, n, X, d)*. Then* X *and* d *are uniquely determined by* τ *and* n*.*

Proof. Let $k_0, ..., k_m$ be the unique sequence of numbers such that if we denote $\Lambda(\phi, k_i)$ by ϕ_i, then $\phi_0 = \phi$, ϕ_{i+1} is an immediate subformula of ϕ_i, and $\phi_m = \psi$. This sequence is uniquely determined by the number n.

During the game that ended in (ψ, n, X, d), the positions (omitting half-rounds) were (ϕ_i, k_i, X_i, d_i), $i = 0, ..., m$. We know that $\phi_0 = \phi$, $X_0 = \{\emptyset\}$ and $d_0 = 1$. If $\phi_i = \neg\phi_{i+1}$, then necessarily $X_{i+1} = X_i$ and $d_{i+1} = 1 - d_i$. If ϕ_{i+1} is a conjunct of ϕ_i, and $d_i = 1$, then $X_{i+1} = X_i$ and $d_{i+1} = 1$. If ϕ_{i+1} is a conjunct of $\phi_i = \psi \wedge \theta$, and $d_i = 0$, then τ determines, on the basis of (ϕ_j, k_j, X_j, d_j), $j = 0, ..., i$, two sets Y and Z such that $X_i = Y \cup Z$. Player **I** chooses whether $X_{i+1} = Y$ or $X_{i+1} = Z$. The result is completely determined by whether $k_{i+1} = k_i + 1$ or $k_{i+1} = k_i + 2 + \text{len}(\psi)$. If $\phi_i = \exists x_n \phi_{i+1}$ and $d_i = 0$, then $X_{i+1} = X_i(M/x_n)$ and $d_{i+1} = d_i$, both uniquely determined by X_i and d_i. If $\phi_i = \exists x_n \phi_{i+1}$ and $d_i = 1$, then τ determines $F : X \longrightarrow M$ on the basis of (ϕ_j, k_j, X_j, d_j), $j = 0, ..., i$. Then $X_{i+1} = X_i(F/x_n)$ and $d_i = 1$, again uniquely determined by τ and X_i.

$\square$

Exercise 48 *Draw the game tree for* $G(\phi)$*, when* ϕ *is* $\neg\exists x_0 P x_0 \wedge \neg\exists x_0 R x_0$*, or* $\exists x_0(P x_0 \wedge R x_0)$*.*

Exercise 49 *Draw the game tree for* $G(\phi)$*, when* ϕ *is* $\forall x_0(P x_0 \vee R x_0)$*, or* $\forall x_0 \exists x_1 (P x_0 \wedge \exists x_2 R x_2 x_1)$*.*

Exercise 50 *Draw the game tree for* $G(\phi)$, *when* ϕ *is* $\exists x_0 \neg P x_0 \rightarrow \forall x_0(P x_0$

Exercise 51 *Let* L *consist of two unary predicates* P *and* R. *Let* $\mathcal{M}$ *be an* L*-structure such that* $M = \{0,1,2,3\}$, $P^{\mathcal{M}} = \{0,1,2\}$, $R^{\mathcal{M}} = \{1,2,3\}$. *Who has a winning strategy in* $G(\phi)$ *if* ϕ *is* $\forall x_0(P x_0 \rightarrow \exists x_1(\neg(x_0 = x_1) \wedge P x_0 \wedge R x_1))$*?*

Exercise 52 *Use the game tree to analyze the formula (3.1).*

Exercise 53 *Suppose* $\mathcal{M}$ *is the binary structure* $(\{0,1,2\}, R)$, *where* R *is as in Figure 3.1. Who has a winning strategy in* $G(\phi)$ *if* ϕ *is* $\forall x_0 \exists x_1(\neg R x_0 x_1 \wedge \forall x_2 \exists x_3(=(x_2, x_3) \wedge \neg R x_2 x_3))$? *Describe the winning strategy.*

Exercise 54 *Suppose* $\mathcal{M}$ *is the binary structure* $(\{0,1,2\}, R)$, *where* R *is as in Figure* **??**. *Who has a winning strategy in* $G(\phi)$ *if* ϕ *is* $\exists x_0 \forall x_1(R x_0 x_1 \vee \exists x_2 \forall x_3(=(x_2, x_3) \rightarrow R x_2 x_3))$? *Describe the winning strategy.*

Exercise 55 *Show that if* τ *is a strategy of player* **II** *in* $G(\phi)$ *and* σ *is a strategy of player* **I** *in* $G(\phi)$, *then there is one and only one play of* $G(\phi)$ *in which player* **II** *has used* τ *and player* **I** *has used* σ. *We denote this play by* $[\tau, \sigma]$.

Exercise 56 *Show that a strategy* τ *of player* **II** *in* $G(\phi)$ *is a winning strategy if and only if player* **II** *wins the play* $[\tau, \sigma]$ *for every strategy* σ *of player I.*

3.3 An Imperfect Information Game for Dependence Logic

The semantics of dependence logic can be defined also by means of a simpler game. In this case, however, we have to put a uniformity restriction on strategies in order to get the correct truth-definition. The restriction has the effect of making the game a game of partial information.

As in the previous section, we pay attention to where a subformula occurs in a formula. This is taken care of by the parameter n in the definition below. It should be borne in mind that conjunctions are assumed to have brackets around them, as in $(\psi \wedge \theta)$. Then if this formula is $\Lambda(\phi, n)$, we can infer that ψ is $\Lambda(\phi, n+1)$ and θ is $\Lambda(\phi, n+2+\mathrm{len}(\psi))$.

Definition 52 *Let ϕ be a sentence of dependence logic. The semantic game $H(\phi)$ in a model $\mathcal{M}$ is the following game: There are two players,* **I** *and* **II**. *A position of the game is a quadruple (ψ, n, s, α), where ψ is $\Lambda(\phi, n)$, s is an assignment the domain of which contains the free variables of ψ, and $\alpha \in \{\mathbf{I}, \mathbf{II}\}$. In the beginning of the game the position is $(\phi, 1, \emptyset, \mathbf{II})$. The rules of the game are as follows:*

1. *The position is $(t_1 = t_2, n, s, \alpha)$: If $t_1^{\mathcal{M}}\langle s\rangle = t_2^{\mathcal{M}}\langle s\rangle$, then player α wins and otherwise the opponent wins.*
2. *The position is $(Rt_1 \ldots t_m, n, s, \alpha)$: If s satisfies $Rt_1 \ldots t_m$ in $\mathcal{M}$, then player α wins, otherwise the opponent wins.*
3. *The position is $({=}(t_1, \ldots, t_m), n, s, \alpha)$: Player α wins.*
4. *The position is $(\neg\phi, n, s, \alpha)$: The game switches to the position $(\phi, n+1, s, \alpha^*)$, where α^* is the opponent of α.*
5. *The position is $(\psi \vee \theta, n, s, \alpha)$: The next position is $(\psi, n+1, s, \alpha)$ or $(\theta, n+2+\mathrm{len}(\psi), s, \alpha)$, and α decides which.*
6. *The position is $(\exists x_m \phi, n, s, \alpha)$: Player α chooses $a \in M$ and the next position is $(\phi, n+2, s(a/x_m), \alpha)$.*

Thus $({=}(t_1, ..., t_n), n, s, \alpha)$ is a safe haven for α. Note that the game is a determined zero-sum game of perfect information. However, we are not really interested in who has a winning strategy in this determined game, but in who has a winning strategy with extra uniformity, as defined below. The uniformity requirement in effect makes the game into a non-determined game of imperfect information.

The concepts of a game tree, play and partial play are defined for this game exactly as for the game $G(\phi)$.

Definition 53 *A* strategy *of player α in $H(\phi)$ is any sequence τ of functions τ_i defined on the set of all partial plays $(p_0, ..., p_{i-1})$ satisfying:*

- *If $p_{i-1} = (\phi \vee \psi, n, s, \alpha)$, then τ tells player α which formula to pick, i.e. $\tau_i(p_0, ..., p_{i-1}) \in \{n+1, n+2+\mathrm{len}(\phi)\}$. If the strategy gives the lower value, player α picks the left-hand formula ϕ, and otherwise the right-hand formula ψ.*
- *If $p_{i-1} = (\exists x_m \phi, n, s, \alpha)$, then τ tells player α which element $a \in M$ to pick i.e. $\tau_i(p_0, ..., p_{i-1}) \in M$.*

We say that player α has used strategy τ in a play of the game $H(\phi)$ if in each relevant case player α has used τ to make his or her choice. More exactly, player α has used τ in a play $p_0, ..., p_n$ if the following two conditions hold for all $i < n$:

- *If $p_{i-1} = (\phi \vee \psi, m, s, \alpha)$ and $\tau_i(p_0, ..., p_{i-1}) = m+1$, then $p_i = (\phi, m+1, s, \alpha)$, while if $\tau_i(p_0, ..., p_{i-1}) = m+2+\mathrm{len}(\phi)$, then $p_i = (\psi, m+2+\mathrm{len}(\phi), s, \alpha)$.*
- *If $p_{i-1} = (\exists x_k \phi, m, s, \alpha)$ and $\tau_i(p_0, ..., p_{i-1}) = a$, then $p_i = (\phi, m+2, s(a/x_k), \alpha)$.*

A strategy of player α in the game $H(\phi)$ is a winning strategy, *if player α wins every play in which she has used the strategy.*

Definition 54 *We call a strategy τ of player* **II** *in the game $H(\phi)$* uniform *if the following condition holds: Suppose $(\Lambda(\phi, m), m, s, \mathbf{II})$ and $(\Lambda(\phi, m), m, s', \mathbf{II})$ are two positions arising in the game when* **II** *has played according to τ. Moreover we assume that $\Lambda(\phi, m)$ is*

$=(t_1, ..., t_n)$. Then if s and s' agree about the values of $t_1, \ldots, t_{n-1}$, they agree about the value of t_n.

Theorem 55 *Suppose ϕ is a sentence of dependence logic. Then $\mathcal{M} \models_{\{\emptyset\}} \phi$ if and only if player* **II** *has a uniform winning strategy in the semantic game $H(\phi)$.*

Proof. Suppose $\mathcal{M} \models_X \phi$. Let τ be a winning strategy of **II** in $G_{\text{place}}(\phi)$. Consider the following strategy of player **II**: She keeps playing $G_{\text{place}}(\phi)$ as an auxiliary game such that if she is in a position (ϕ, n, X, d) in $G_{\text{place}}(\phi)$ and has just moved in the semantic game, then:

($\star$) Suppose the position is (ϕ, n, s, α). Then **II** is in a position (ϕ, n, X, d), playing τ, in $G_{\text{place}}(\phi)$ and $s \in X$. If $\alpha =$ **II** then $d = 1$. If $\alpha =$ **I** then $d = 0$.

Let us check that **II** can actually follow this strategy and win. In the beginning $\mathcal{M} \models_{\{\emptyset\}} \phi$, so ($\star$) holds.

1. ϕ is $t_1 = t_2$ or $Rt_1 \ldots t_n$. If $\alpha =$ **II**, s satisfies ϕ in $\mathcal{M}$. So **II** wins. If $\alpha =$ **I**, s does not satisfy ϕ in $\mathcal{M}$, and again **II** wins.

2. ϕ is $=(t_1, ..., t_n)$. If $\alpha =$ **II**, then **II** wins by definition. On the other hand $s \in X$, so $X \neq \emptyset$, and we must have $\alpha =$ **II**.

3. ϕ is $\neg\psi$ and the position in $G_{\text{place}}(\phi)$ is $(\neg\psi, n, X, d)$. By the rules of $G_{\text{place}}(\phi)$, the next position is $(\psi, n+1, X, 1-d)$. So the game can proceed to position $(\psi, n+1, s, \alpha^*)$ and **II** maintains ($\star$).

4. ϕ is $(\psi \vee \theta)$ and the position in $G_{\text{place}}(\phi)$ is $((\psi \vee \theta), n, X, d)$. Suppose $\alpha =$ **I** and $d = 0$. Then both $(\psi, n+1, X, d)$ and $(\theta, n+2+\text{len}(\psi), X, d)$ are possible positions in $G_{\text{place}}(\phi)$ while **II** uses τ. The next position is $(\psi, n+1, s, 0)$ or $(\theta, n+2+\text{len}(\psi_0), s, 0)$, and **I** chooses which. Condition ($\star$) remains valid, whichever she has to hold. Suppose then $\alpha =$ **II** and $d = 1$. Strategy τ gives X_0 and X_1 such that $X = X_0 \cup X_1$ and **II** wins with τ both in the position

$(\psi, n+1, X_0, 1)$ and in $(\theta, n+2+\mathrm{len}(\psi_0), X_1, 1)$. Since $s \in X$, we have either $s \in X_0$ or $s \in X_1$. Let us say $s \in X_0$. We let **I** play ψ in $G_{\text{place}}(\phi)$. The game $G_{\text{place}}(\phi)$ proceeds to $(\psi, n+1, X_0, 1)$. We let **II** play in $H(\phi)$ the sentence ψ. Condition $(\star)$ remains valid. The situation is similar if $s \in X_1$.

5. ϕ is $\exists x\psi$. We leave this as an exercise.

We claim that the strategy is uniform. Suppose s and s' are assignments arising from the game when **II** plays the above strategy and the game ends in the same dependence formula $=(t_1, ..., t_n)$. Let the ending positions be $(\Lambda(\phi, n), s, \alpha)$ and $(\Lambda(\phi, n), s', \alpha)$. Since **II** wins, $\alpha = \mathbf{II}$. When the games ended, player **II** had reached the position $(=(t_1, ..., t_n), n, X, 1)$ on one hand and the position $(=(t_1, ..., t_n), n, X', 1)$ on the other hand in $G_{\text{place}}(\phi)$ playing τ. By Lemma 51, $X = X'$. Suppose s and s' agree about the values of $t_1, \ldots, t_{n-1}$. Since **II** wins in the position $(=(t_1, ..., t_n), n, X, 1)$ and $s, s' \in X$, it follows that s and s' agree about the value of t_n. This strategy gives one direction of the theorem.

For the other direction, suppose player **II** has a uniform winning strategy τ in the semantic game starting from $(\phi, 1, \emptyset, \mathbf{II})$. Let X_n be the set of s such that $(\Lambda(\phi, n), n, s, \alpha)$ is the position in some play where **II** used τ. Note that α depends only on n, so we can denote it by α_n. We show by induction on subformulas $\Lambda(\phi, n)$ of ϕ that $(\Lambda(\phi, n), X_n, d_n) \in \mathcal{T}$, where $d_n = 1$ if and only if $\alpha_n = \mathbf{II}$. Putting $n = 1$ we get $\alpha_1 = \mathbf{II}$ and we get the desired result.

1. Suppose $\Lambda(\phi, n)$ is $t_1 = t_2$ or $R(t_1, \ldots, t_n)$. We show that the quadruple $(\Lambda(\phi, n), n, X_n, d)$ is in $\mathcal{T}$. Let $s \in X_n$. Let the quadruple $(\Lambda(\phi, n), n, s, \alpha_n)$ be a position in some play where **II** used τ. Since **II** wins with τ, $(\Lambda(\phi, n), X_n, d) \in \mathcal{T}$.

2. Suppose $\Lambda(\phi, n)$ is $=(t_1, ..., t_n)$. Suppose first $\alpha_n = \mathbf{II}$. Suppose s and s' are in X_n and agree about the values of $t_1, \ldots, t_{n-1}$. By

the definition of X_n, $(\Lambda(\phi,n),n,s,\mathbf{II})$ and $(\Lambda(\phi,n),n,s',\mathbf{II})$ are positions in some plays where **II** used τ. Since τ is uniform, s and s' agree about the value of t_n. The case $\alpha_n=\mathbf{I}$ cannot occur since τ is a winning strategy.

3. Suppose $\Lambda(\phi,n)$ is $\neg\psi$. Note that $X_n=X_{n+1}$. By the induction hypothesis, $(\psi,X_n,1-d)\in\mathcal{T}$, hence $(\neg\psi,X_n,d)\in\mathcal{T}$.

4. Suppose $\Lambda(\phi,n)$ is $(\psi\vee\theta)$. Suppose first $\alpha_n=\mathbf{I}$. Then both $(\Lambda(\phi,n{+}1),n{+}1,s,\mathbf{I})$ and $(\Lambda(\phi,n{+}2{+}\mathrm{len}(\psi)),n{+}2{+}\mathrm{len}(\psi),s,\mathbf{I})$ can be a positions in some plays where **II** has used τ. By the induction hypothesis, $(\psi,X_{n+1},0)\in\mathcal{T}$ and $(\theta,X_{n+2+\mathrm{len}(\psi)},0)\in\mathcal{T}$. Note that $X_n\subseteq X_{n+1}\cap X_{n+2+\mathrm{len}(\psi)}$. Hence $(\psi\vee\theta,X_n,0)\in\mathcal{T}$. Suppose then $\alpha_n=\mathbf{II}$. Now $X=Y\cup Z$, where Y is the set of $s\in X_n$ such that $(\Lambda(\phi,n+1),n+1,s,\alpha_n)$ and Z is the set of $s\in X_n$ such that $(\Lambda(\phi,n{+}2{+}\mathrm{len}(\psi)),n{+}2{+}\mathrm{len}(\psi),s,\alpha_n)$ By the induction hypothesis, $(\psi,X_{n+1},1)\in\mathcal{T}$ and $(\theta,X_{n+2+\mathrm{len}(\psi)},1)\in\mathcal{T}$. Hence $(\psi\wedge\theta,X_n,1)\in\mathcal{T}$.

5. Suppose $\Lambda(\phi,n)$ is $\exists x\psi$. We leave this as an exercise.

$\square$

Exercise 57 *Draw the game tree for $H(\phi)$, when ϕ is* $(\neg\exists x_0{=}(x_0)\vee\neg\exists x_0R$

Exercise 58 *Draw the game tree for $H(\phi)$, when ϕ is* $\forall x_0\exists x_1({=}(x_1)\vee Rx_0)$.

Exercise 59 *Draw the game tree for $H(\phi)$, when ϕ is* $\exists x_0(\neg Px_0\rightarrow\forall x_1({=}(x$

Exercise 60 *Let L consist of two unary predicates P and R. Let $\mathcal{M}$ be an L-structure such that $M=\{0,1,2,3\}$, $P^{\mathcal{M}}=\{0,1,2\}$ and $R^{\mathcal{M}}=\{1,2,3\}$. Who has a winning strategy in $H(\phi)$ if ϕ is* $\exists x_0({=}(x_0)\wedge Rx_0)$*? Describe the winning strategy.*

Exercise 61 *Let $\mathcal{M}$ be as in Exercise 60. Does* **II** *have a uniform winning strategy in $H(\phi)$ if ϕ is the sentence* $\forall x_0(Px_0\rightarrow\exists x_1({=}(x_0,x_1)\wedge\neg($

Exercise 62 *Let $\mathcal{M}$ be as in Exercise 60. Does II have a uniform winning strategy in $H(\phi)$ if ϕ is the sentence* $\exists x_0(Px_0 \wedge \forall x_1(=(x_0, x_1) \vee (x_0$

Exercise 63 *Suppose $\mathcal{M}$ is the binary structure $(\{0, 1, 2\}, R)$, where R is as in Figure* **??**. *Does* **II** *have a uniform winning strategy in $H(\phi)$ if ϕ is* $\forall x_0 \exists x_1(\neg Rx_0x_1 \wedge \forall x_2 \exists x_3(=(x_2, x_3) \wedge \neg Rx_2x_3))$?

Exercise 64 *Suppose $\mathcal{M}$ is the binary structure $(\{0, 1, 2\}, R)$, where R is as in Figure* **??**. *Does* **II** *have a uniform winning strategy in $H(\phi)$ if ϕ is* $\exists x_0 \forall x_1(Rx_0x_1 \vee \exists x_2 \forall x_3(=(x_2, x_3) \rightarrow Rx_2x_3))$?

Exercise 65 *Show that neither of the below two winning strategies of player* **II** *in $H(\phi)$ is uniform, when ϕ is the sentence* $\forall x_0 \exists x_1((=(x_1) \wedge x_0 = x_1) \vee (=(x_1) \wedge x_0 = x_1))$ *and the universe is* $\{0, 1, 2\}$.

x_0	x_1	$\vee$
0	0	*left*
1	1	*left*
2	2	*right*

x_0	x_1	$\vee$
0	0	*left*
1	1	*right*
2	2	*right*

Exercise 66 *Which of the below two strategies of player* **II** *in $H(\phi)$ can be completed so that the strategy becomes a uniform winning strategy of* **II**? *Here ϕ is the sentence* $\forall x_0 \exists x_1((=(x_1) \wedge \neg x_0 = x_1) \vee (=(x_1) \wedge \neg x_0 = x_1)))$ *and the universe is* $\{0, 1, 2\}$.

x_0	x_1	$\vee$
0	2	*left*
1	2	
2	0	*right*

x_0	x_1	$\vee$
0	1	*left*
1	2	
2	0	*right*

Chapter 4

Model Theory

Many model theoretic results for dependence logic can be proved by means of a reduction to existential second order logic. We establish this reduction in the first section of this chapter. This gives immediately such results as the Compactness Theorem, the Löwenheim-Skolem Theorem, and the Craig Interpolation Theorem.

4.1 From $\mathcal{D}$ to Σ_1^1

We associate with every formula ϕ of dependence logic a second order sentence which is in a sense equivalent to ϕ. This is in fact nothing more than a formalization of the truth definition of ϕ (Definition 5). What is interesting is that the second order sentence, which we denote by $\tau_{1,\phi}(S)$, is not just any second order sentence but a particularly simple second order existential sentence, called a Σ_1^1-sentence. Such sentences have a close relationship with first order logic, especially on countable models. It turns out that their relationship with dependence logic is even closer. In a sense they are one and the same thing. It is the main purpose of this section to explain exactly what is this sense in which they are one and the same thing.

Theorem 56 *We can associate with every formula $\phi(x_{i_1}, ..., x_{i_n})$ of $\mathcal{D}$ in vocabulary L and every $d \in \{0, 1\}$ a Σ_1^1-sentence $\tau_{d,\phi}(S)$, where S is n-ary, such that for all L-structures $\mathcal{M}$ and teams X with $\mathrm{dom}(X) = \{x_{i_1}, ..., x_{i_n}\}$ the following are equivalent*

1. $(\phi, X, d) \in \mathcal{T}$
2. $(\mathcal{M}, X) \models \tau_{d,\phi}(S)$.

Proof. We modify the approach of [13, Section 3] to fit our setup. The sentence $\tau_{d,\phi}(S)$ is simply Definition 5 written in another way. There is nothing new in $\tau_{d,\phi}(S)$, and in each case the proof of the claimed equivalence is straightforward (see Exercises **??** and **??**).

Case 1: Suppose $\phi(x_{i_1}, ..., x_{i_n})$ is $t_1 = t_2$ or $Rt_1...t_n$. We rewrite (D1), (D2), (D5) and (D6) of Definition 5 by letting $\tau_{1,\phi}(S)$ be $\forall x_{i_1}...\forall x_{i_n}(Sx_{i_1}...x_{i_n} \rightarrow \phi(x_{i_1}, ..., x_{i_n}))$ and by letting $\tau_{0,\phi}(S)$ be $\forall x_{i_1}...\forall x_{i_n}(Sx_{i_1}...x_{i_n} \rightarrow \neg\phi(x_{i_1}, ..., x_{i_n}))$

Case 2:

Suppose $\phi(x_{i_1}, ..., x_{i_n})$ is the dependence formula $=(t_1(x_{i_1}, ..., x_{i_n}), ..., t_m(x_{i_1}, ..., x_{i_n}))$, where $i_1 < ... < i_n$. Recall conditions (D3) and (D4) of Definition 5. Following these conditions, we define $\tau_{1,\phi}(S)$ as follows:

Subcase 2.1: $m = 0$. We let $\tau_{1,\phi}(S) = \top$ and $\tau_{0,\phi}(S) = \neg\top$.

Subcase 2.2: $m = 1$. Now $\phi(x_{i_1}, ..., x_{i_n})$ is the dependence formula $=(t_1(x_{i_1}, ..., x_{i_n}))$. We let $\tau_{1,\phi}(S)$ be the formula

$$\begin{aligned}\forall x_{i_1}...\forall x_{i_n}\forall x_{i_n+1}...\forall x_{i_n+n}((&Sx_{i_1}...x_{i_n} \wedge Sx_{i_n+1}...x_{i_n+n})\\ &\rightarrow t_1(x_{i_1}, ..., x_{i_n}) = t_1(x_{i_n+1}, ..., x_{i_n+n}))\end{aligned}$$

and we further let $\tau_{0,\phi}(S)$ be the formula $\forall x_{i_1}...\forall x_{i_n}\neg Sx_{i_1}...x_{i_n}$.

Subcase 2.3: If $m > 1$ we let $\tau_{1,\phi}(S)$ be the formula

$$\begin{aligned}\forall x_{i_1}...\forall x_{i_n}\forall x_{i_n+1}...\forall x_{i_n+n}((&Sx_{i_1}...x_{i_n} \wedge Sx_{i_n+1}...x_{i_n+n} \wedge\\ &t_1(x_{i_1}, ..., x_{i_n}) = t_1(x_{i_n+1}, ..., x_{i_n+n}) \wedge\\ &...\\ &t_{m-1}(x_{i_1}, ..., x_{i_n}) = t_{m-1}(x_{i_n+1}, ..., x_{i_n+n}))\\ &\rightarrow t_m(x_{i_1}, ..., x_{i_n}) = t_m(x_{i_n+1}, ..., x_{i_n+n}))\end{aligned}$$

and we further let $\tau_{0,\phi}(S)$ be the formula $\forall x_{i_1}...\forall x_{i_n}\neg Sx_{i_1}...x_{i_n}$.

Case 3: Suppose $\phi(x_{i_1}, ..., x_{i_n})$ is the disjunction

$(\psi(x_{j_1}, ..., x_{j_p}) \vee \theta(x_{k_1}, ..., x_{k_q}))$, where $\{i_1, ..., i_n\} = \{j_1, ..., j_p\} \cup \{k_1, ..., k_q\}$. We let the sentence $\tau_{1,\phi}(S)$ be

$$\begin{aligned}&\exists R \exists T(\tau_{1,\psi}(R) \wedge \tau_{1,\theta}(T) \wedge \\ &\qquad \forall x_{i_1} ... \forall x_{i_n}(Sx_{i_1} ... x_{i_n} \rightarrow (Rx_{j_1} ... x_{j_p} \vee Tx_{k_1} ... x_{k_q})))\end{aligned}$$

and we let the sentence $\tau_{0,\phi}(S)$ be

$$\begin{aligned}&\exists R \exists T(\tau_{0,\psi}(R) \wedge \tau_{0,\theta}(T) \wedge \\ &\qquad \forall x_{i_1} ... \forall x_{i_n}(Sx_{i_1} ... x_{i_n} \rightarrow (Rx_{j_1} ... x_{j_p} \wedge Tx_{k_1} ... x_{k_q}))).\end{aligned}$$

.

Case 4: ϕ is $\neg\psi$. $\tau_{d,\phi}(S)$ is the formula $\tau_{1-d,\psi}(S)$.

Case 5: Suppose $\phi(x_{i_1}, ..., x_{i_n})$ is the formula $\exists x_{i_{n+1}} \psi(x_{i_1}, ..., x_{i_{n+1}})$. $\tau_{1,\phi}(S)$ is the formula

$$\exists R(\tau_{1,\psi}(R) \wedge \forall x_{i_1} ... \forall x_{i_n}(Sx_{i_1} ... x_{i_n} \rightarrow \exists x_{i_{n+1}} Rx_{i_1} ... x_{i_{n+1}}))$$

and $\tau_{0,\phi}(S)$ is the formula

$$\exists R(\tau_{0,\psi}(R) \wedge \forall x_{i_1} ... \forall x_{i_n}(Sx_{i_1} ... x_{i_n} \rightarrow \forall x_{i_{n+1}} Rx_{i_1} ... x_{i_{n+1}})).$$

$\square$

$\exists f \forall x \forall y \phi(x, y, f(x, y), f(y, x))$

$\forall x \forall y \exists z \forall x' \forall y' \exists z'$

$(= (x', y', z') \wedge$

$((x = x' \wedge y = y') \rightarrow z = z') \wedge$

$((x = y' \wedge x' = y) \rightarrow \phi(x, y, z, z'))$

Corollary 57 *For every sentence ϕ of $\mathcal{D}$ there are Σ^1_1-sentences $\tau_{1,\phi}$ and $\tau_{0,\phi}$ such that for all models $\mathcal{M}$ we have $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models \tau_{1,\phi}$.$\mathcal{M} \models \neg\phi$ if and only if $\mathcal{M} \models \tau_{0,\phi}$.*

Proof. Let $\tau_{d,\phi}$ be the result of replacing in $\tau_{d,\phi}(S)$ every occurrence of the 0-ary relation symbol S by $\top$. Now the claim follows from Theorem 56. $\square$

Exercise 67 *Write down* $\tau_{1,\exists x_1 =(x_1)}(S)$ *and* $\tau_{0,\exists x_1 =(x_1)}(S)$.

Exercise 68 *What is* $\tau_{1,\phi}(S)$ *if* ϕ *is* $\exists x_1(=(x_1) \vee x_1 = x_0)$*?*

Exercise 69 *What is* $\tau_{1,\phi}(S)$ *if* ϕ *is:* $\forall x_0 \exists x_1 \forall x_2 \exists x_3(=(x_2, x_3) \wedge \neg(x_1 = x_4)$

Exercise 70 *Show that if* ϕ *is* Σ^1_1, $\mathcal{M} \models \phi$ *and* $\mathcal{M} \cong \mathcal{N}$, *then* $\mathcal{N} \models \phi$.

4.2 Applications of Σ^1_1

The Σ^1_1-representation of $\mathcal{D}$-formulas yields some immediate but all the same very important applications. They are all based on model theoretic properties of first order logic, which we now review:

Compactness Theorem of first order logic: Suppose T is an arbitrary set of sentences of first order logic such that every finite subset of T has a model. Then T itself has a model.

Löwenheim-Skolem Theorem of first order logic: Suppose ϕ is a sentence of first order logic such that ϕ has an infinite model or arbitrarily large finite models. Then ϕ has models of all infinite cardinalities.

Craig Interpolation Theorem of first order logic: Suppose ϕ and ψ are sentences of first order logic such that $\models \phi \to \psi$. Suppose the vocabulary of ϕ is L_ϕ and that of ψ is L_ψ. Then there is a first order sentence θ of vocabulary $L_\phi \cap L_\psi$ such that $\models \phi \to \theta$ and $\models \theta \to \psi$.

We can now easily derive similar results for dependence logic by appealing to the Σ^1_1-representation of $\mathcal{D}$-sentences:

Theorem 58 (Compactness Theorem of $\mathcal{D}$) *Suppose* Γ *is an arbitrary set of sentences of dependence logic such that every finite subset of* Γ *has a model. Then* Γ *itself has a model.*

Proof. Let $\Gamma = \{\phi_i : i \in I\}$ and let L be the vocabulary of Γ. Let $\tau_{1,\phi_i} = \exists S^i_1 \ldots \exists S^i_{n_i} \psi_i$, where ψ_i is first order. By changing symbols we can assume all S^i_j are different symbols. Let T be the first order theory $\{\psi_i : i \in I\}$ in the vocabulary $L' = L \cup \{S^i_j : i \in I, 1 \leq j \leq n_i\}$. Every finite subset of T has a model. By the Compactness Theorem of first order logic there is an L'-structure $\mathcal{M}'$ that is a model of the theory T itself. The reduction $\mathcal{M} = \mathcal{M}' \restriction L$ of $\mathcal{M}'$ to the original vocabulary L is, by definition, a model of Γ. $\square$

Theorem 59 (Löwenheim-Skolem Theorem of $\mathcal{D}$) *Suppose ϕ is a sentence of dependence logic such that ϕ either has an infinite model or has arbitrarily large finite models. Then ϕ has models of all infinite cardinalities, in particular, ϕ has a countable model and an uncountable model.*

Proof. Let $\tau_{1,\phi} = \exists S_1 ... \exists S_n \psi$, where ψ is first order in the vocabulary $L' = L \cup \{S_1, ..., S_n\}$. Suppose κ is an arbitrary infinite cardinal number. By the Löwenheim-Skolem Theorem of first order logic there is an L'-model $\mathcal{M}'$ of ψ of cardinality κ. The reduction $\mathcal{M} = \mathcal{M}' \restriction L$ of $\mathcal{M}'$ to the original vocabulary L is a model of ϕ of cardinality κ. $\square$

Corollary 60 ([21]) *A sentence of dependence logic in the empty vocabulary is true in one infinite model (or arbitrarily large finite ones) if and only it is true in every infinite model.*

Proof. All models of the empty vocabulary of the same cardinality are isomorphic. Thus the claim follows from Theorem 59. $\square$

We shall address the Craig Interpolation Theorem later and derive first a Separation Theorem which is an equivalent formulation in the case of first order logic.

Theorem 61 (Separation Theorem) *Suppose ϕ and ψ are sentences of dependence logic such that ϕ and ψ have no models in common. Let the vocabulary of ϕ be L and the vocabulary of ψ be L'. Then there is a sentence θ of $\mathcal{D}$ in the vocabulary $L \cap L'$ such* *that every model of ϕ is a model of θ, but θ and ψ have no models in common. In fact, θ can be chosen to be first order.*

Proof. Let $\tau_{1,\phi} = \exists S_1 ... \exists S_n \phi_0$, where ϕ_0 is first order in the vocabulary L_0. Let $\tau_{1,\psi} = \exists S'_1 ... \exists S'_m \psi_0$, where ψ_0 is first order in the vocabulary L'_0. Without loss of generality, $\{S_1, ..., S_n\} \cap \{S'_1, ..., S'_m\} = \emptyset$. Note that $\models \phi_0 \rightarrow \neg\psi_0$ for if $\mathcal{M}$ is a model of $\phi_0 \wedge \psi_0$, then $\mathcal{M} \restriction L \models \phi \wedge \psi$, contrary to the assumption that ϕ and ψ have no models in common. By the Craig Interpolation Theorem for first order logic there is a first order sentence θ of vocabulary $L \cap L'$ such that $\models \phi_0 \rightarrow \theta$ and $\models \theta \rightarrow \neg\psi_0$. Every model of ϕ is a model of θ, but θ and ψ have no models in common. $\square$

A particularly striking application of Theorem 61 is the following special case in which ϕ and ψ not only have no models in common but furthermore every model satisfies one of them:

Theorem 62 (Failure of the Law of Excluded Middle) *Suppose ϕ and ψ are sentences of dependence logic such that for all models $\mathcal{M}$ we have $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \not\models \psi$. Then ϕ is logically equivalent to a first order sentence θ such that ψ is logically equivalent to $\neg\theta$.*

Proof. The first order θ obtained in the proof of Theorem 61 is the θ we seek. $\square$

Note that it is perfectly possible to have for all *finite* models $\mathcal{M}$ $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \not\models \psi$ without ϕ or ψ being logically equivalent to a first order sentence. For example, in the empty vocabulary ϕ can say the size of the universe is even while ψ says it is odd.

Definition 63 *A sentence ϕ of dependence logic is called* determined in $\mathcal{M}$ *if $\mathcal{M} \models \phi$ or $\mathcal{M} \models \neg\phi$. Otherwise ϕ is called* non-determined in $\mathcal{M}$. *We say that ϕ is* determined *if ϕ is determined in every structure.*

A typical non-determined sentence (from [1]) is $\forall x_0 \exists x_1 (=(x_1) \wedge x_0 = x_1)$, which is non-determined in every structure with at least two elements. The following corollary shows that it is not at all difficult to find other non-determined sentences.

Corollary 64 *Every determined sentence of dependence logic is strongly logically equivalent to a first order sentence.*

Proof. Suppose ϕ is determined. Thus for all $\mathcal{M}$ we have $\mathcal{M} \models \phi$ or $\mathcal{M} \models \neg\phi$. It follows that for all $\mathcal{M}$ we have $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \not\models \neg\phi$. By Theorem 62 there is a first order θ such that ϕ is logically equivalent to θ and $\neg\phi$ is logically equivalent to $\neg\theta$. Thus ϕ is strongly logically equivalent to θ. $\square$

Thus we can take any sentence of dependence logic, which is not strongly equivalent to a first order sentence, and we know that there are models in which the sentence is non-determined.

Example 65 *The sentence $\Phi_{\rm wf}$ is non-determined in every infinite well-ordered structure. This can be seen either by a direct argument based on the truth definition, or by the following indirect argument: Suppose $\mathcal{M}$ is an infinite well-ordered linear order in which $\Phi_{\rm wf}$ is determined. Thus $\mathcal{M} \models \neg\Phi_{\rm wf}$. Let Γ be the following set of sentences of dependence logic: $\neg\Phi_{\rm wf}, c_1 < c_0, c_2 < c_1, ..., c_{n+1} < c_n, ...$ It is evident that every finite subset of Γ is true in an expansion of $\mathcal{M}$. By the Compactness Theorem there is a model $\mathcal{M}'$ of the whole Γ. Then $\mathcal{M}'$ is ill-founded and therefore satisfies $\Phi_{\rm wf}$. This contradicts the fact that $\mathcal{M}'$ also satisfies $\neg\Phi_{\rm wf}$.*

For more examples of non-determinacy, see [28].

Exercise 71 *Show that the sentence $\Phi_{\rm wf}$ is non-determined in all sufficiently big finite linear orders.*

Exercise 72 *Show that $\Phi_{\rm even}$ is non-determined in every sufficiently large finite model of odd size.*

Exercise 73 *Show that Φ_∞ is non-determined in every sufficiently big finite model.*

Exercise 74 *Show that Φ_{cmpl} is non-determined in every complete dense linear order.*

Exercise 75 *Suppose ϕ_n, $n \in \mathbb{N}$, are sentences of $\mathcal{D}$ such that each ϕ_n is true in some model, and moreover $\phi_{n+1} \Rightarrow \phi_n$ for all n. Show that there is one model $\mathcal{M}$ in which each ϕ_n is true.*

Exercise 76 *Show that if ϕ is a sentence of $\mathcal{D}$ and ψ is a first order sentence such that every countable*[1] *model of ϕ is a model of ψ, then every model of ϕ is a model of ψ.*

Exercise 77 *Give a sentence ϕ of $\mathcal{D}$ and a first order sentence ψ such that every countable model of ψ is a model of ϕ and vice versa, but some model of ψ is not a model of ϕ.*

4.3 From Σ_1^1 to $\mathcal{D}$

We have seen that representing formulas of dependence logic in Σ_1^1 form is a powerful method for getting model theoretic results about dependence logic. We now show that this method is in a sense the best possible. Namely, we can also translate any Σ_1^1-sentence back to dependence logic.

We prove first a fundamental property of first order and Σ_1^1 formulas. Its various formulations all carry the name of Thoralf Skolem ([24] (see [25])), who proved the below result already in 1920. The basic idea is that the existential second order quantifiers in front of Σ_1^1-formulas are so powerful that they subsume all other existential quantifiers.

Theorem 66 (Skolem Normal Form Theorem) *Every Σ_1^1 formula ϕ is logically equivalent to an existential second order formula*

$$\exists f_1 \ldots \exists f_n \forall x_1 \ldots \forall x_m \psi, \tag{4.1}$$

[1] i.e. countably infinite or finite

where ψ is quantifier free and $f_1, \ldots, f_n$ are function symbols. The formula (4.1) is called a Skolem Normal Form of ϕ.

Corollary 67 (Skolem Normal Form Theorem for $\mathcal{D}$) *For every formula ϕ of dependence logic and every $d \in \{0,1\}$ there is a Σ^1_1-sentence $\tau^*_{d,\phi}(S)$ of the form*

$$\exists f_1 \ldots \exists f_n \forall x_1 \ldots \forall x_m \psi, \tag{4.2}$$

where ψ is quantifier free, such that the following are equivalent:

1. $(\phi, X, d) \in \mathcal{T}$
2. $(\mathcal{M}, X) \models \tau^*_{d,\phi}(S)$

In particular, for every sentence ϕ of dependence logic there are Σ^1_1-sentences $\tau^*_{1,\phi}$ and $\tau^*_{0,\phi}$ of the form (4.2) such that for all models $\mathcal{M}$ we have $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models \tau^*_{1,\phi}$.$\mathcal{M} \models \neg\phi$ if and only if $\mathcal{M} \models \tau^*_{0,\phi}$.

The above corollary gives an easy proof of the Löwenheim-Skolem Theorem for dependence logic (Theorem 59). Namely, suppose ϕ is a given sentence of dependence logic with an infinite model or arbitrarily large finite models. By compactness we may assume ϕ indeed has an infinite model $\mathcal{M}$. Let $\tau^*_{1,\phi}$ be of the form $\exists f_1 \ldots \exists f_n \forall x_1 \ldots \forall x_m \psi$. Thus there are interpretations $f_i^{\mathcal{M}'}$ of the function symbols f_i in an expansion of $\mathcal{M}'$ of $\mathcal{M}$ such that $\mathcal{M}'$ satisfies $\forall x_1 \ldots \forall x_m \psi$. Let N be a countable subset of M such that N is closed under all the n functions $f_i^{\mathcal{M}'}$. Because $\forall x_1 \ldots \forall x_m \psi$ is universal, it is still true in the countable substructure $\mathcal{M}^*$ of $\mathcal{M}$ generated by N. Thus $\mathcal{M}^*$ is a countable model of ϕ. This is in line with the original proof of Skolem. If we wanted a model of size κ for a given infinite cardinal number κ, the argument would be similar but we would first use compactness to get a model of size at least κ.

Theorem 68 ([4],[30]) *For every Σ_1^1-sentence ϕ there is a sentence ϕ^* in dependence logic such that for all $\mathcal{M}$: $\mathcal{M} \models \phi \iff \mathcal{M} \models \phi^*$.*

Proof. We may assume ϕ is of the form

$$\Phi = \exists f_1 \ldots \exists f_n \forall x_1 \ldots \forall x_m \psi, \tag{4.3}$$

where ψ is quantifier free. We will perform some reductions on (4.3) in order to make it more suitable for the construction of ϕ.

Step 1: If ψ contains nesting of the function symbols $f_1, \ldots, f_n$ or of the function symbols of the vocabulary, we can remove them one by one by using the equivalence of $\models \psi(f_i(t_1, \ldots, t_m))$ and $\forall x_1 \ldots \forall x_m((t_1 = x_1 \ \wedge \ldots \wedge \ t_m = x_m) \rightarrow \psi(f_i(x_1, \ldots, x_m)))$. Thus we may assume that all terms occurring in ψ are of the form x_i or $f_i(x_{i_1}, \ldots, x_{i_k})$.

Step 2: If ψ contains an occurrence of a function symbol $f_i(x_{i_1}, \ldots, x_{i_k})$ with the same variable occurring twice, e.g. $i_s = i_r$, $1 < r < k$, we can remove such by means of a new variable x_l and the equivalence

$$\begin{aligned} \models \ & \forall x_1 \ldots \forall x_m \psi(f_i(x_{i_1}, \ldots, x_{i_k})) \leftrightarrow \\ & \forall x_1 \ldots \forall x_m \forall x_l (x_l = x_r \rightarrow \psi(f_i(x_{i_1}, \ldots, x_{i_{r-1}}, x_l, x_{i_{r+1}}, \ldots, x_{i_k}))) \end{aligned}$$

Thus we may assume that if a term such as $f_i(x_{i_1}, \ldots, x_{i_k})$ occurs in ψ, its variables are all distinct.

Step 3: If ψ contains two occurrences of the same function symbol but with different variables or with the same variables in different order, we can remove such using appropriate equivalences. If $\{i_1, ..., i_k\} \cap \{j_1, ..., j_k\} = \emptyset$, we have the equivalence

$$\begin{aligned} \models \ & \forall x_1 \ldots \forall x_m \psi(f_i(x_{i_1}, \ldots, x_{i_k}), f_i(x_{j_1}, \ldots, x_{j_k})) \leftrightarrow \\ & \exists f_i' \forall x_1 \ldots \forall x_m (\psi(f_i(x_{i_1}, \ldots, x_{i_k}), f_i'(x_{j_1}, \ldots, x_{j_k})) \wedge \\ & ((x_{i_1} = x_{j_1} \ \wedge \ldots \wedge \ x_{i_k} = x_{j_k}) \rightarrow \\ & f_i(x_{i_1}, \ldots, x_{i_k}) = f_i'(x_{j_1}, \ldots, x_{j_k}))) \end{aligned}$$

We can reduce the more general case, where $\{i_1, ..., i_k\} \cap \{j_1, ..., j_k\} \neq \emptyset$, to this case by introducing new variables, as in Step 2. (We are grateful to Ville Nurmi for pointing out the necessity of this.) Thus we may assume that for each function symbol f_i occurring in ψ there are $j^i_1, \ldots, j^i_{n_i}$ such that *all* occurrences of f_i are of the form $f_i(x_{j^i_1}, \ldots, x_{j^i_{m_i}})$ and $j^i_1, \ldots, j^i_{m_i}$ are all different from each other.

In sum we may assume the function terms that occur in ψ are of the form $f_i(x_{j^i_1}, \ldots, x_{j^i_{m_i}})$ and for each i the variables $x_{j^i_1}, \ldots, x_{j^i_{m_i}}$ and their order is the same. Let N be greater than all the $x_{j^i_k}$. Following the notation of (4.3), let ϕ^* be the sentence

$$\begin{array}{ll} \forall x_1 \ldots \forall x_m \exists x_{N+1} \ldots \exists x_{N+n} & (=(x_{j^1_1}, \ldots, x_{j^1_{m_1}}, x_{N+1}) \wedge \\ & \ldots \\ & (=(x_{j^n_1}, \ldots, x_{j^n_{m_n}}, x_{N+n}) \wedge \psi') \end{array}$$

where ψ' is obtained from ψ by replacing everywhere $f_i(x_{j^i_1}, \ldots, x_{j^i_{m_i}})$ by x_{N+i}. This is clearly the desired sentence. $\square$

It is noteworthy that the $\mathcal{D}$-representation of a given Σ^1_1-sentence given above is of universal-existential form, that is, of the form $\forall x_{n_1} ... \forall x_{n_k} \exists x_{m_1} ... \exists x_{m_l} \psi$ where ψ is quantifier-free. Moreover, ψ is just a conjunction of dependence statements $=(x_1, ..., x_n)$ and a quantifier-free first order formula. This is a powerful *normal form* for dependence logic.

Corollary 69 *For any sentence ϕ of dependence logic of vocabulary L and for every $L' \subseteq L$ there is a sentence ϕ' of dependence logic of vocabulary L' such that the following are equivalent:*

1. *$\mathcal{M} \models \phi'$.*
2. *There is an expansion $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N} \models \phi$.*

Proof. Let ψ be a Σ^1_1-sentence logically equivalent with ϕ. We assume for simplicity that $L \setminus L'$ consists of just one predicate symbol R. Let

ϕ' be a sentence of dependence logic logically equivalent with the Σ^1_1-sentence $\exists R\psi$. Then ϕ' is a sentence satisfying the equivalence of the conditions 1 and 2. $\Box$

Corollary 69 implies in a trivial way the

Corollary 70 (Uniform Interpolation Property) *Suppose ϕ is a sentence of $\mathcal{D}$. Let L be the vocabulary of ϕ. For every $L' \subseteq L$ there is a sentence ϕ' of $\mathcal{D}$ in the vocabulary L' which is a* uniform interpolant *of ϕ in the following sense: $\phi \Rightarrow \phi'$ and if ψ is a sentence of $\mathcal{D}$ in a vocabulary L'' such that $\phi \Rightarrow \psi$ and $L \cap L'' = L'$, then $\phi' \Rightarrow \psi$.*

Proof. Let ϕ' be as in Corollary 69. By its very definition, ϕ' is a logical consequence of ϕ. Suppose then ψ is a sentence of $\mathcal{D}$ in a vocabulary L'' such that $\phi \Rightarrow \psi$ and $L \cap L'' = L'$. If $\mathcal{M}''$ is an L''-structure which is a model of ϕ', then $\mathcal{M}''\restriction L'$ is a model of ϕ', whence there is an expansion $\mathcal{M}$ of $\mathcal{M}''\restriction L'$ to a model of ϕ. Since $\phi \Rightarrow \psi$, $\mathcal{M}$ is a model of ψ. But $\mathcal{M}\restriction L'' = \mathcal{M}''\restriction L''$. Thus $\mathcal{M}'' \models \psi$. $\Box$

For a version of the *Beth Definability Theorem*, see Exercise 86.

Exercise 78 *Give a Skolem Normal Form for the following first order formulas* $\forall x_0 \exists x_1 x_0 = x_1$, $\exists x_0 \forall x_1 \neg x_0 = x_1$, $(\exists x_0 P x_0 \vee \forall x_0 \neg P x_0)$.

Exercise 79 *Give a Skolem Normal Form for the following first order formula* $\forall x_0 \exists x_1 \forall x_2 \exists x_3 ((x_0 = x_4 \rightarrow \neg(x_1 = x_4)) \wedge (x_0 < x_3 \leftrightarrow x_1 < x_2)))$.

Exercise 80 *Write the following Σ^1_1-sentences in Skolem Normal Form:*

1. $\exists x_0 \exists f \forall x_1 (\neg f x_1 = x_0 \wedge \forall x_2 (f x_0 = f x_1 \rightarrow x_0 = x_1))$.
2. $\exists R(\forall x_0 \forall x_1 \forall x_2 ((R x_0 x_1 \wedge R x_1 x_2) \rightarrow R x_0 x_2)$
   $\wedge \forall x_0 \forall x_1 (R x_0 x_1 \vee R x_1 x_0 \vee x_0 = x_1)$
   $\wedge \forall x_0 \neg R x_0 x_0 \wedge \forall x_0 \exists x_1 R x_0 x_1)$.

Exercise 81 *Give a sentence of $\mathcal{D}$ which is logically equivalent to the following Σ^1_1-sentence in Skolem Normal Form:*

1. $\exists f_0 \exists f_1 \forall x_0 \forall x_1 \phi(x_0, x_1, f_0(x_0, x_1), f_1(x_0, x_1))$.
2. $\exists f_0 \exists f_1 \forall x_0 \forall x_1 \phi(x_0, x_1, f_0(x_0, x_1), f_1(x_1))$.
3. $\exists f_0 \exists f_1 \forall x_0 \phi(x_0, f_0(x_0), f_1(x_0))$.

In each case ϕ is quantifier-free and first order.

Exercise 82 *Give a sentence of $\mathcal{D}$ which is logically equivalent to the Σ^1_1-sentence*

$$\exists f \forall x_0 \forall x_1 \phi(x_0, x_1, f(x_0, x_1), f(x_1, x_0)),$$

where ϕ is quantifier-free.

Exercise 83 *Express the Henkin quantifier* $\begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} R(x, y, u, v) \leftrightarrow$ $\exists f \exists g \forall x \forall u R(x, f(x), u, g(u))$ *by a formula of dependence logic.*

Exercise 84 *Suppose $\mathcal{M}$ is an L-structure and $P \subseteq M^n$. We say that P is $\mathcal{D}$-*definable in $\mathcal{M}$ *if there is a sentence $\phi(c_1, ..., c_n)$ of $\mathcal{D}$ with new constant symbols $c_1, ..., c_n$ such that the following are equivalent for all $a_1, ..., a_n \in M$:*

1. $(a_1, ..., a_n) \in P$
2. $(\mathcal{M}, a_1, ..., a_n) \models \phi$,

where $(\mathcal{M}, a_1, ..., a_n)$ denotes the expansion of $\mathcal{M}$ obtained by interpreting c_i in $(\mathcal{M}, a_1, ..., a_n)$ by a_i. We then say that ϕ defines *P in $\mathcal{M}$. Show that if P and Q are $\mathcal{D}$- definable, then so are $P \cap Q$ and $P \cup Q$, but not necessarily $P \setminus Q$.*

Exercise 85 *Recall the definition of $\mathcal{D}$-definability in a model in Exercise 84. Let L be a vocabulary. Suppose ψ is a $\mathcal{D}$-sentence in a vocabulary $L \cup \{R\}$, where R is a new n-ary predicate symbol. We say that R is $\mathcal{D}$-*definable in models of ψ, *if there is a $\mathcal{D}$-sentence ϕ*

of vocabulary $L \cup \{c_1, ..., c_n\}$ such that ϕ defines R in every model of ψ. Prove the following useful criterion for $\mathcal{D}$-undefinability: If ψ has two models $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M}\lceil L = \mathcal{N}\lceil L$ but $R^{\mathcal{M}} \neq R^{\mathcal{N}}$, then R is not $\mathcal{D}$-definable in models of ψ. (In the case of first order logic this is known as the Padoa Principle.*)*

Exercise 86 *Recall the definition of $\mathcal{D}$-definability in models of a sentence in Exercise 85. Let L be a vocabulary. Suppose ψ is a $\mathcal{D}$-sentence in a vocabulary $L \cup \{R\}$, where R is a new n-ary predicate symbol. Suppose any two models $\mathcal{M}$ and $\mathcal{N}$ of ψ such that $\mathcal{M}\lceil L = \mathcal{N}\lceil L$ satisfy also $R^{\mathcal{M}} = R^{\mathcal{N}}$. Show that R is $\mathcal{D}$-definable in models of ψ (In the case of first order logic this is known as the* Beth Definability Theorem.*)*

Exercise 87 *[2] Suppose ϕ and ψ are $\mathcal{D}$-sentences such that no model satisfies both ϕ and ψ. Show that there is a sentence θ of $\mathcal{D}$ such that $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models \theta$ and $\mathcal{M} \models \psi$ if and only if $\mathcal{M} \models \neg\theta$.*

4.4 Truth-Definitions

In 1933 the Polish logician Alfred Tarski defined the concept of truth in a general setting (see e.g. [26]) and pointed out what is known as Tarski's Undefinability of Truth argument: no language can define its own truth, owing to the *Liar Paradox*, namely to the sentence "This sentence is false." This sentence is neither true nor false, contrary to the Law of Excluded Middle, which Tarski took for granted. Already 1931 the Austrian logician Kurt Gödel (see [7]), working not on arbitrary formalized languages but on first order number theory, had constructed, using a lengthy process referred to as the arithmetization of syntax, the sentence "This sentence is unprovable." This sentence cannot be provable, for then it would be true, hence unprovable. So it is unprovable and hence true. Its negation cannot be provable either, for else the negation would be true. So it is an example of a sentence which is *independent* of first order number theory. This is known as Gödel's First Incompleteness Theorem. Gödel's technique could be used to make exact sense of undefinability of truth (see below) and to prove it exactly for first order number theory.

Exercise 88 *Consider "If this sentence is true, then its negation is true." Derive a contradiction.*

Exercise 89 *Consider "It is not true that this sentence is true." Derive a contradiction.*

Exercise 90 *([19]) Consider the sentences*

(1) *It is raining in Warsaw.*

(2) *It is raining in Vienna.*

(3) *Exactly one the sentences (1)-(3) is true.*

Under what kind of weather conditions in Europe is sentence (3) paradoxical?

4.4.1 Undefinability of Truth

To even formulate the concept of definability of truth we have to introduce a method for speaking about a formal language in the language itself. The clearest way of doing this is by means of Gödel-numbering. Each sentence ϕ is associated with a natural number $\ulcorner\phi\urcorner$, its *Gödel-number,* in a systematic way, described in section 4.4.2. Moreover, we assume that our language has a name $\underline{n}$ for each natural number n.

Definition 71 *A* truth-definition *for any model $\mathcal{M}$ and any formal language $\mathcal{L}$, such as first order logic or dependence logic, is a formula $\tau(x_0)$ of some possibly other formal language $\mathcal{L}'$ such that for each sentence ϕ of $\mathcal{L}$ we have*

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{M} \models \tau(\underline{\ulcorner\phi\urcorner}). \tag{4.4}$$

A stronger requirement would be $\mathcal{M} \models \phi \leftrightarrow \tau(\underline{\ulcorner\phi\urcorner})$, but this would be true only in the presence of the Law of Excluded Middle, as $(\phi \leftrightarrow \psi) \Rightarrow (\phi \vee \neg\phi)$. An even stronger requirement would be the provability of $\phi \leftrightarrow \tau(\underline{\ulcorner\phi\urcorner})$ from some axioms, but we abandon this, too, in the current setup.

By the vocabulary $L_{\{+,\times\}}$ of arithmetic we mean a vocabulary appropriate for the study of number theory, with a symbol N for the set of natural numbers. We specify $L_{\{+,\times\}}$ in detail below. We call an L-structure $\mathcal{M}_\omega$, where $L \supseteq L_{\{+,\times\}}$, "a model of Peano's axioms" if the reduct of $\mathcal{M}_\omega$ to the vocabulary $\{N, +, \times\}$ satisfies the first order Peano axioms of number theory. The results on definability of truth are relevant even if we assume that $N^{\mathcal{M}_\omega}$ is the whole universe of the model $\mathcal{M}_\omega$. Below, $\mathcal{M}_\omega$ denotes such a model of Peano's axioms.

Theorem 72 (Gödel's Fixed Point Theorem) *For any first order formula $\phi(x_0)$ in the vocabulary of arithmetic there is a first order sentence ψ of the same vocabulary such that for all models $\mathcal{M}_\omega$ of Peano's axioms, $\mathcal{M}_\omega \models \psi$ if and only if $\mathcal{M}_\omega \models \phi(\underline{\ulcorner\psi\urcorner})$.*

Proof. Let Sub be the set of triples $\langle \ulcorner w \urcorner, \ulcorner w' \urcorner, n \rangle$ where w' is obtained from w by replacing x_0 by the term $\underline{n}$. Since recursive relations are representable in models of Peano's axioms, there is a first order formula $\sigma(x_0, x_1, x_2)$ such that

$$\langle n, m, k \rangle \in Sub \iff \mathcal{M}_\omega \models \sigma(\underline{n}, \underline{m}, \underline{k}).$$

W.l.o.g. x_0 is not bound in $\sigma(x_0, x_1, x_2)$ and x_0 and x_1 are not bound in $\phi(x_0)$. Let $\theta(x_0)$ be the formula $\exists x_1(\phi(x_1) \wedge \sigma(x_0, x_1, x_0))$. Let $k = \ulcorner\theta(x_0)\urcorner$ and $\psi = \theta(\underline{k})$. Then $\mathcal{M}_\omega \models \psi$ if and only if $\mathcal{M}_\omega \models \phi(\underline{\ulcorner\psi\urcorner})$. $\square$

The above result does not hold just for first order logic but for any extension of first order logic the syntax of which is sufficiently effectively given, for example dependence logic.

Theorem 73 (Tarski's Undefinability of Truth Result) *First order logic does not have a truth-definition in first order logic for any model $\mathcal{M}_\omega$ of Peano's axioms.*

Proof. Let $\tau(x_0)$ be as in Definition 71. By Theorem 72 there is a sentence ψ such that

$$\mathcal{M}_\omega \models \psi \text{ if and only if } \mathcal{M}_\omega \models \neg\tau(\underline{\ulcorner\psi\urcorner}). \tag{4.5}$$

If $\mathcal{M}_\omega \models \psi$, then $\mathcal{M}_\omega \models \tau(\underline{\ulcorner\psi\urcorner})$ by (4.4), and $\mathcal{M}_\omega \models \neg\tau(\underline{\ulcorner\psi\urcorner})$ by (4.5). Hence $\mathcal{M}_\omega \not\models \psi$. Now $\mathcal{M}_\omega \not\models \tau(\underline{\ulcorner\psi\urcorner})$ by (4.4), and $\mathcal{M}_\omega \not\models \neg\tau(\underline{\ulcorner\psi\urcorner})$ by (4.5). So neither $\tau(\underline{\ulcorner\psi\urcorner})$ nor $\neg\tau(\underline{\ulcorner\psi\urcorner})$ is true in $\mathcal{M}_\omega$. This contradicts the Law of Excluded Middle, which says in this case $\mathcal{M}_\omega \models (\tau(\underline{\ulcorner\psi\urcorner}) \vee \neg\tau(\underline{\ulcorner\psi\urcorner}))$. $\Box$

Theorem 73 has many stronger formulations. As the almost trivial proof above shows, no language for which the Gödel Fixed Point Theorem can be proved and which satisfies the Law of Excluded Middle for its negation can have a truth-definition in the language itself. We shall not elaborate more on this point here, as our topic, dependence logic, certainly does not satisfy the Law of Excluded Middle for its negation.

Exercise 91 *Prove $\mathcal{M}_\omega \models \psi$ if and only if $\mathcal{M}_\omega \models \phi(\underline{\ulcorner\psi\urcorner})$ in the proof of Theorem 72.*

4.4.2 Definability of Truth in First Order Logic

We turn to another important contribution of Tarski, namely that truth is *implicitly* (or even better – *inductively*) definable in first order logic. In dependence logic the implicit definition can even be turned into an explicit definition by means of Theorem 68, as emphasized by Hintikka [10]. So after all, truth *is* definable, albeit only implicitly. The realization of this may be an even more important contribution of Tarski to logic than the undefinability of truth.

We shall carry out in some detail the definition of truth for first order logic. We shall omit many details, as these are well covered by the literature. For simplicity, we assume the vocabulary $L_{\{+,\times\}}$ of arithmetic includes all the machinery needed for arithmetization. All that we really need is a pairing function, but the pursuit of such minimalism is not relevant for the main argument and belongs to other contexts. Another simplification is that we only consider truth in models $\mathcal{M}_\omega$ of Peano's axioms.

We consider a finite vocabulary $L = \{c_1, ..., c_n, R_1, ..., R_m, f_1, ..., f_k\}$ containing $L_{\{+,\times\}}$. When we specify below what $L_{\{+,\times\}}$ should contain we assume they are all among $c_1, ..., c_n, R_1, ..., R_m, f_1, ..., f_k$. Let $r_i = \#(R_i)$. If $w = w_0...w_k$ is a string of symbols in the alphabet $=, c_i, R_i, f_i, (,), \neg, \vee, \exists$, the Gödel-number $\ulcorner w \urcorner$ of w is the natural number $\ulcorner w \urcorner = {p_0}^{\#(w_0)+1} \cdot \ldots \cdot {p_k}^{\#(w_k)+1}$ where $p_0, p_1, ...$ are the prime numbers in increasing order and

$\#(=) = 0$	$\#(() = 1$	$\#()) = 2$	$\#(\neg) = 3$
$\#(\vee) = 4$	$\#(\wedge) = 5$	$\#(\exists) = 6$	$\#(\forall) = 7$
$\#(c_i) = 4 + 4i$	$\#(x_i) = 5 + 4i$	$\#(R_i) = 6 + 4i$	$\#(f_i) = 7 + 4i$

The vocabulary $L_{\{+,\times\}}$ has a symbol $\underline{0}$ for zero, a symbol $\underline{1}$ for one, and the names $\underline{n}$ of the other natural numbers n are defined inductively as terms $+\underline{n}1$. We assume $L_{\{+,\times\}}$ has the following symbols used to represent syntactic operations:

Symbol	Interpretation in $\mathcal{M}_\omega$
POS-ID$x_0x_1x_2$	x_0 is $\ulcorner t = t' \urcorner$, where $x_1 = \ulcorner t \urcorner$ and $x_2 = \ulcorner t' \urcorner$
NEG-ID$x_0x_1x_2$	x_0 is $\ulcorner \neg t = t' \urcorner$, where $x_1 = \ulcorner t \urcorner$ and $x_2 = \ulcorner t' \urcorner$
POS-ATOM$_i x_0x_1...x_{r_i}$	x_0 is $\ulcorner R_i t_1 ... t_{r_i} \urcorner$, where $x_1 = \ulcorner t_1 \urcorner$,..., $x_{r_i} = \ulcorner t_{r_i} \urcorner$
NEG-ATOM$_i x_0x_1...x_{r_i}$	x_0 is $\ulcorner \neg R_i t_1 ... t_{r_i} \urcorner$, where $x_1 = \ulcorner t_1 \urcorner$,..., $x_{r_i} = \ulcorner t_{r_i} \urcorner$
CONJ$x_0x_1x_2$	x_0 is $\ulcorner (\phi \wedge \psi) \urcorner$, where $x_1 = \ulcorner \phi \urcorner$ and $x_2 = \ulcorner \psi \urcorner$
DISJ$x_0x_1x_2$	x_0 is $\ulcorner (\phi \vee \psi) \urcorner$, where $x_1 = \ulcorner \phi \urcorner$ and $x_2 = \ulcorner \psi \urcorner$
EXI$x_0x_1x_2$	x_0 is $\ulcorner \exists x_n \phi \urcorner$, where $x_1 = n$ and $x_2 = \ulcorner \phi \urcorner$
UNI$x_0x_1x_2$	x_0 is $\ulcorner \forall x_n \phi \urcorner$, where $x_1 = n$ and $x_2 = \ulcorner \phi \urcorner$

We assume that among the symbols of $L_{\{+,\times\}}$ are functions that provide a bijection between elements of the model and finite sequences of elements of the model[2]. Thus it makes sense to interpret arbitrary elements of $\mathcal{M}_\omega$ as assignments. We assume $L_{\{+,\times\}}$ has also the following symbols: (We think of x_0 as an assignment below):

TRUE-ID$x_0x_1x_2$	x_0 satisfies the identity $t = t'$, where $x_1 = \ulcorner t \urcorner$ and $x_2 = \ulcorner t' \urcorner$
FALSE-ID$x_0x_1x_2$	x_0 satisfies the non-identity $\neg t = t'$, where $x_1 = \ulcorner t \urcorner$ and $x_2 = \ulcorner t' \urcorner$
TRUE-ATOM$_i x_0x_1...x_{r_i}$	x_0 satisfies $R_i t_1...t_{r_i}$, where $x_1 = \ulcorner t_1 \urcorner$,...,$x_{r_i} = \ulcorner t_{r_i} \urcorner$
FALSE-ATOM$_i x_0x_1...x_{r_i}$	x_0 satisfies $\neg R_i t_1...t_{r_i}$, where $x_1 = \ulcorner t_1 \urcorner$,...,$x_{r_i} = \ulcorner t_{r_i} \urcorner$
AGR$x_0x_1x_2$	x_0 and x_2 are assignments that agree about variables other than x_1

All the above symbols are easily definable in terms of + and $\cdot$ in first order logic any model $\mathcal{M}_\omega$ of Peano's axioms, if wanted. Now we take a new predicate symbol SAT, not to be included in $L_{\{+,\times\}}$, (and not to be definable in terms of + and $\cdot$ in first order logic) with the intuitive meaning:

SATx_0x_1	x_0 is an assignment s and x_1 is $\ulcorner \phi \urcorner$ for some L-formula ϕ such that $\mathcal{M}_\omega \models_s \phi$.

The point is that SAT is (implicitly) definable in terms of the others by the first order sentence θ_L as follows:

$$
\begin{aligned}
&\forall x_0 \forall x_1 (\mathrm{SAT} x_0 x_1 \leftrightarrow \\
&\quad \exists x_2 \exists x_3 (\text{POS-ID} x_1 x_2 x_3 \wedge \text{TRUE-ID} x_0 x_2 x_3) \vee \\
&\quad \exists x_2 \exists x_3 (\text{NEG-ID} x_1 x_2 x_3 \wedge \text{FALSE-ID} x_0 x_2 x_3) \vee \\
&\quad \exists x_2 ... \exists x_{r_1+1} (\text{POS-ATOM}_1 x_1 x_2 ... x_{r_1+1} \wedge \text{TRUE-ATOM}_1 x_0 x_2 ... x_{r_1+1}) \vee \\
&\quad \ldots \\
&\quad \exists x_2 ... \exists x_{r_m+1} (\text{POS-ATOM}_m x_1 x_2 ... x_{r_m+1} \wedge \text{TRUE-ATOM}_m x_0 x_2 ... x_{r_m+1}) \vee \\
&\quad \exists x_2 ... \exists x_{r_1+1} (\text{NEG-ATOM}_1 x_1 x_2 ... x_{r_1+1} \wedge \text{FALSE-ATOM}_1 x_0 x_2 ... x_{r_1+1}) \vee \\
&\quad \ldots \\
&\quad \exists x_2 ... \exists x_{r_m+1} (\text{NEG-ATOM}_m x_1 x_2 ... x_{r_m+1} \wedge \text{FALSE-ATOM}_m x_0 x_2 ... x_{r_m+1}) \vee \\
&\quad \exists x_2 \exists x_3 (\mathrm{CONJ} x_1 x_2 x_3 \wedge (\mathrm{SAT} x_0 x_2 \wedge \mathrm{SAT} x_0 x_3)) \vee \\
&\quad \exists x_2 \exists x_3 (\mathrm{DISJ} x_1 x_2 x_3 \wedge (\mathrm{SAT} x_0 x_2 \vee \mathrm{SAT} x_0 x_3)) \vee \\
&\quad \exists x_2 \exists x_3 (\mathrm{EXI} x_1 x_2 x_3 \wedge \exists x_4 (\mathrm{AGR} x_0 x_2 x_4 \wedge \mathrm{SAT} x_4 x_3)) \vee \\
&\quad \exists x_2 \exists x_3 (\mathrm{UNI} x_1 x_2 x_3 \wedge \forall x_4 (\mathrm{AGR} x_0 x_2 x_4 \rightarrow \mathrm{SAT} x_4 x_3)))
\end{aligned}
$$

The implicit (or inductive) nature of this definition (known as Tarski's Truth Definition) manifests itself in the fact that SAT appears on both sides of the equivalence sign in θ_L, and only positively in each case. There is no guarantee that θ really fixes what SAT is (see e.g. [17]). However, for the part of the actual formulas, or their representatives in $\mathcal{M}_\omega$, the set SAT is unique.

[2] In the special case that $N^{\mathcal{M}_\omega}$ (i.e. $\mathbb{N}$) is the whole universe of $\mathcal{M}_\omega$ the encoding of finite sequences of elements of the model by elements of the model can be achieved by means of the unique factorization of integers, or alternatively by means of the Chinese Remainder Theorem.

Theorem 74 *If the L-structure $\mathcal{M}_\omega$ is a model of Peano's axioms, then:*

1. *$(\mathcal{M}_\omega, Sat_{\mathbb{N}}) \models \theta_L$, where $Sat_{\mathbb{N}}$ is the set of pairs $\langle s, \underline{\ulcorner \phi \urcorner}^{\mathcal{M}_\omega} \rangle$ such that $\mathcal{M}_\omega \models_s \phi$.*

2. *If $(\mathcal{M}_\omega, S) \models \theta_L$ and $(\mathcal{M}_\omega, S') \models \theta_L$, then $S \cap Sat_{\mathbb{N}} = S' \cap Sat_{\mathbb{N}}$.*

Proof. Claim (i) is tedious but trivial, assuming that our concepts are correctly defined. The claim we prove is the second one. To this end suppose $(\mathcal{M}_\omega, S) \models \theta_L$ and $(\mathcal{M}_\omega, S') \models \theta_L$. We prove

$$\langle s, \underline{\ulcorner \phi \urcorner}^{\mathcal{M}_\omega} \rangle \in S \text{ if and only if } \mathcal{M}_\omega \models_s \phi \tag{4.6}$$

for all ϕ. Since the same holds for S' by symmetry, we get the desired result.

Case 1: ϕ is of the form $t = t'$, $\neg t = t'$, $R_i t_1 ... t_n$ or $\neg R_i t_1 ... t_n$. Claim (4.6) follows from our interpretation of POS-ID, NEG-ID, POS-ATOM. NEG-ATOM, TRUE-ID, FALSE-ID, TRUE-ATOM$_i$, and FALSE-ATOM$_i$ in $\mathcal{M}_\omega$.

Case 2: ϕ is of the form $(\phi_0 \wedge \phi_1)$. Claim (4.6) follows from the conjunction part of the definition of θ_L and our interpretation of CONJ in $\mathcal{M}_\omega$.

Case 3: ϕ is of the form $(\phi_0 \vee \phi_1)$. Claim (4.6) follows from the disjunction part of the definition of θ_L and our interpretation of DISJ in $\mathcal{M}_\omega$.

Case 4: ϕ is of the form $\exists x_n \psi$. Claim (4.6) follows from the part of the existential quantifier of the definition of θ_L and our interpretation of AGR and EXI in $\mathcal{M}_\omega$.

Case 5: ϕ is of the form $\forall x_n \psi$. Claim (4.6) follows from the part of the universal quantifier of the definition of θ_L and our interpretation of AGR and UNI in $\mathcal{M}_\omega$. $\square$

By means of the satisfaction relation SAT we can define truth by means of the formula: $\mathrm{TRUE}(x_0) = \exists x_1 \mathrm{SAT} x_1 x_0$.

Corollary 75 *The following are equivalent for all first order sentences ϕ in the vocabulary L and any model $\mathcal{M}_\omega$ of Peano's axiom:*

1. $\mathcal{M}_\omega \models \phi$.
2. $(\mathcal{M}_\omega, S) \models (\theta_L \wedge TRUE(\underline{\ulcorner\phi\urcorner}))$ *for some* $S \subseteq \mathbb{N}^2$.
3. $(\mathcal{M}_\omega, S) \models (\theta_L \rightarrow TRUE(\underline{\ulcorner\phi\urcorner}))$ *for all* $S \subseteq \mathbb{N}^2$.

Let us call a model $\mathcal{M}_\omega$ of Peano's axioms *standard* if the interpretation of the predicate N of the vocabulary of arithmetic in $\mathcal{M}_\omega$ is the set of natural numbers, the interpretation of $+$ is the addition of natural numbers, and the interpretation of $\times$ is the multiplication of natural numbers.

Corollary 76 *Suppose $\mathcal{M}_\omega$ is any standard model of Peano's axioms. If $(\mathcal{M}_\omega, S) \models \theta_L$ and $(\mathcal{M}_\omega, S') \models \theta_L$, then $S = S'$.*

First order definable relations on standard $\mathcal{M}_\omega$ are called *arithmetical*. The definition of truth given by Corollary 75 is not first order, so we cannot say truth is arithmetical. A definition that has both existential second order and universal second order definition, as truth in the above corollary, is called *hyperarithmetical*. So we can say first order truth on $\mathcal{M}_\omega$ is not arithmetical but hyperarithmetical. For more on hyperarithmetical definitions see [22].

Exercise 92 *Show that we cannot remove "standard" from Corollary 76.*

4.4.3 Definability of Truth in $\mathcal{D}$

We now move from first order logic back to dependence logic. We observed in Corollary 75 that the truth definition of first order logic on a model $\mathcal{M}_\omega$ of Peano's axioms can be given in first order logic if one existential second order quantifier is allowed. In dependence logic we can express the existential second order quantifier and thus the truth definition of first order logic on $\mathcal{M}_\omega$ can be given in dependence logic. This can be extended to a truth definition of all of dependence logic and that is our goal in this section.

The fact that Σ^1_1 has a truth definition in Σ^1_1 on a structure with enough coding is well-known in descriptive set theory. It was observed in [10] that, as an application of Theorem 68, we have a truth definition for $\mathcal{D}$ in $\mathcal{D}$ on any structure with enough coding.

We shall consider below a vocabulary $L \supseteq L_{\{+,\times\}} \cup \{c\}$, where c is a new constant symbol. If ϕ is a sentence of $\mathcal{D}$ in this vocabulary, we indicate the inclusion of a new constant by writing ϕ as $\phi(c)$. Then if d is another constant symbol, then $\phi(d)$ is the sentence obtained from $\phi(c)$ by replacing c everywhere by d.

Theorem 77 ([10]) *Suppose $L \supseteq L_{\{+,\times\}} \cup \{c\}$ is finite. There is a sentence $\tau(c)$ of $\mathcal{D}$ in the vocabulary L such that for all sentences ϕ of $\mathcal{D}$ in the vocabulary L and all models $\mathcal{M}_\omega$ of Peano's axioms: $\mathcal{M}_\omega \models \phi$ if and only if $\mathcal{M}_\omega \models \tau(\underline{\ulcorner\phi\urcorner})$.*

Proof. By Theorem 56 every sentence of $\mathcal{D}$ is logically equivalent to a Σ^1_1-sentence of the form

$$\exists R_1 \ldots \exists R_n \phi^*, \tag{4.7}$$

where ϕ^* is first order. We now replace the second order quantifiers $\exists R_1 \ldots \exists R_n$ by just one second order quantifier $\exists R_0$, where R_0 is a unary predicate symbol not in L. At the same time we replace every occurrence of $R_i t_1 \ldots t_n$ in ϕ^* by $R_0\langle \underline{i}, t_1, \ldots, t_n\rangle$, where $(a_1, \ldots, a_n) \mapsto \langle a_1, \ldots, a_n\rangle$ is the function in $L_{\{+,\times\}}$ coding n-sequences. Let the result be ϕ^{**}. Now the following are equivalent for any ϕ in $\mathcal{D}$ in vocabulary L and any L-structure $\mathcal{M}_\omega$, which is a model of Peano's axioms:

(i) $\mathcal{M}_\omega \models \phi$.

(ii) $(\mathcal{M}_\omega, Z) \models \phi^{**}$ for some $Z \subseteq \mathbb{N}$.

Let $L' = L \cup \{R_0\}$. By Corollary 75, we obtain the equivalence of (ii) with

(iii) $(\mathcal{M}_\omega, Z, Sat) \models (\theta_{L'} \wedge \mathrm{TRUE}(\underline{\ulcorner \phi^{**} \urcorner}))$ for some $Sat \subseteq \mathbb{N}^2$ and some $Z \subseteq \mathbb{N}$.

By Corollary 69 there is a sentence $\tau_0(c)$ of vocabulary L such that (iii) is equivalent to

(iv) $\mathcal{M}_\omega \models \tau_0(\underline{\ulcorner \phi^{**} \urcorner})$.

Let $t(x_0)$ be a term of the vocabulary $L_{\{+,\times\}}$ such that for all sentences ϕ of $\mathcal{D}$ of vocabulary L the value of $t(\underline{\ulcorner \phi \urcorner})$ in any $L_{\{+,\times\}}$-model of Peano's axioms is $\ulcorner \phi^{**} \urcorner$. Let $\tau(c)$ be the L-sentence $\tau_0(t(c))$. Then (i)-(iv) are equivalent to

(v) $\mathcal{M}_\omega \models \tau(\underline{\ulcorner \phi \urcorner})$.

□

Now that we have constructed the truth definition by recourse to Σ^1_1-sentences, it should be pointed out that we could write $\tau(c)$ also directly in $\mathcal{D}$, imitating the inductive definition of truth given in Definition 5. This is the approach of [14].

Let us now go back to the Liar Paradox. By Theorem 72 there is a sentence λ of $\mathcal{D}$ in the vocabulary $L_{\{+,\times\}}$ such that for all $\mathcal{M}_\omega$ $\mathcal{M}_\omega \models \lambda$ if and only if $\mathcal{M}_\omega \models \neg\tau(\underline{\ulcorner \lambda \urcorner})$. Intuitively, λ says "This sentence is not true." By Theorem 77, $\mathcal{M}_\omega \models \lambda$ if and only if $\mathcal{M}_\omega \models \tau(\underline{\ulcorner \lambda \urcorner})$. Thus $\mathcal{M}_\omega \models \tau(\underline{\ulcorner \lambda \urcorner})$ if and only if $\mathcal{M}_\omega \models \neg\tau(\underline{\ulcorner \lambda \urcorner})$, which is, of course, only possible if $\mathcal{M}_\omega \not\models \tau(\underline{\ulcorner \lambda \urcorner})$ and $\mathcal{M}_\omega \not\models \neg\tau(\underline{\ulcorner \lambda \urcorner})$. Still another way of putting this is: $\mathcal{M}_\omega \not\models \tau(\underline{\ulcorner \lambda \urcorner}) \vee \neg\tau(\underline{\ulcorner \lambda \urcorner})$. Thus we get the pleasing result that the assertion that the Liar sentence is true (in the sense of $\tau(c)$) is non-determined. This is in harmony with the intuition that the Liar sentence does not have a truth value.

Exercise 93 *Suppose* $\mathcal{M}_\omega \models \psi_0$ *if and only if* $\mathcal{M}_\omega \models \tau(\underline{\ulcorner \psi_1 \urcorner})$ *and* $\mathcal{M}_\omega \models \psi_1$ *if and only if* $\mathcal{M}_\omega \models \neg\tau(\underline{\ulcorner \psi_0 \urcorner})$. *Show that* $\tau(\underline{\ulcorner \psi_0 \urcorner})$ *and* $\tau(\underline{\ulcorner \psi_1 \urcorner})$ *are non-determined.*

Exercise 94 *Contemplation of the sentence "This sentence is not true" leads immediately to the paradox* $(\star)$*: the sentence is true if and only if it is not true. We have seen that we can write a sentence* λ *with the meaning "This sentence is not true" in* $\mathcal{D}$*. Still we do not get the result that* $(\lambda \leftrightarrow \neg\lambda)$ *is true. Indeed, show that* $(\phi \leftrightarrow \neg\phi)$ *has no models, whatever* ϕ *in* $\mathcal{D}$ *is. Explain why the existence of* λ *does not lead to the paradox* $(\star)$*.*

Exercise 95 *Suppose* ψ *says "If* ψ *is true, then* $\neg\psi$ *is true," i.e.* $\mathcal{M}_\omega \models \psi$ *if and only if* $\mathcal{M}_\omega \models (\tau(\underline{\ulcorner\psi\urcorner}) \rightarrow \tau(\underline{\ulcorner\neg\psi\urcorner}))$*. Show that* ψ *is non-determined.*

Exercise 96 *Suppose* ψ *says "It is not true that* ψ *is true." i.e.* $\mathcal{M}_\omega \models \psi$ *if and only if* $\mathcal{M}_\omega \models \neg\tau(\underline{\ulcorner\tau(\underline{\ulcorner\psi\urcorner})\urcorner})$*. Show that* $\tau(\underline{\ulcorner\tau(\underline{\ulcorner\psi\urcorner})\urcorner})$ *is non-determined.*

Exercise 97 *Show that there cannot be* $\tau'(c)$ *in* $\mathcal{D}$ *such that for all* ϕ *and all* $\mathcal{M}_\omega$ *we have* $\mathcal{M}_\omega \not\models \phi$ *if and only if* $\mathcal{M}_\omega \models \tau'(\underline{\ulcorner\phi\urcorner})$*.*

Exercise 98 *[[23]] Suppose* $\mathcal{M}_\omega$ *is a model of Peano's axioms, as above. Let* T *be the set of sentences of* $\mathcal{D}$ *that are true in* $\mathcal{M}_\omega$*. Show that there is a model* $\mathcal{M}$ *of* T *such that for some* $a \in M$ *we have* $\mathcal{M} \models (\tau(a) \wedge \tau(\neg a))$*.*

4.5 The Ehrenfeucht-Fraïssé Game for Dependence Logic

We define the concept of elementary equivalence of models and give this concept a game characterization. In this section we assume vocabularies do not contain function symbols, for simplicity.

Definition 78 *Two models* $\mathcal{M}$ *and* $\mathcal{N}$ *of the same vocabulary are* $\mathcal{D}$-equivalent*, in symbols* $\mathcal{M} \equiv_{\mathcal{D}} \mathcal{N}$ *if they satisfy the same sentences of dependence logic.*

It is easy to see that isomorphic structures are $\mathcal{D}$-equivalent and that $\mathcal{D}$-equivalence is an equivalence relation. However, despite the quite strong Löwenheim-Skolem Theorem of dependence logic we have the following negative result about $\mathcal{D}$-equivalence.

Proposition 79 *There is an uncountable model $\mathcal{M}$ in a finite vocabulary such that $\mathcal{M}$ is not $\mathcal{D}$-equivalent to any countable models.*

Proof. Let $\mathcal{M} = (\mathbb{R}, \mathbb{N})$ be a model for the vocabulary $\{P\}$. Suppose $\mathcal{M} \equiv_{\mathcal{D}} \mathcal{N}$, where $\mathcal{N}$ is countable. It ic clear that there is a one-to-one function from N into $P^{\mathcal{N}}$. Let ϕ be a sentence of $\mathcal{D}$ which says exactly this. Since $\mathcal{N} \models \phi$, we have $\mathcal{M} \models \phi$, a contradiction. $\Box$

It turns out that the following more basic concept is a far better concept to start with:

Definition 80 *Suppose $\mathcal{M}$ and $\mathcal{N}$ are structures for the same vocabulary. We say that $\mathcal{M}$ is $\mathcal{D}$-*semiequivalent* to $\mathcal{N}$, in symbols $\mathcal{M} \Rrightarrow_{\mathcal{D}} \mathcal{N}$ if $\mathcal{N}$ satisfies every sentence of dependence logic that is true in $\mathcal{M}$.*

Note that equivalence and semiequivalence are equivalent concepts if dependence logic is replaced by first order logic.

Proposition 81 *Every infinite model in a countable vocabulary is $\mathcal{D}$-semiequivalent to models of all infinite cardinalities.*

Proof. Suppose $\mathcal{M}$ is an infinite model with a countable vocabulary L. Let T be the set of sentences of dependence logic in the vocabulary L that are true in $\mathcal{M}$. By Theorem 59 the theory T has models of all infinite cardinalities (Theorem 59 is formulated for sentences only but the proof for countable theories is the same). $\Box$

We now introduce an Ehrenfeucht-Fraïssé game adequate for dependence logic and use this game to characterize $\Rrightarrow_{\mathcal{D}}$.

Definition 82 *Let $\mathcal{M}$ and $\mathcal{N}$ be two structures of the same vocabulary. The game EF_n has two players and n moves. The position after move m is a pair (X, Y), where $X \subseteq M^m$ and $Y \subseteq N^m$ for* *some m. In the beginning the position is $(\{\emptyset\},\{\emptyset\})$ and $i_0 = 0$. Suppose the position after move number m is (X,Y). There are the following possibilities for the continuation of the game:*

Splitting move: *Player I represents X as a union $X = X_0 \cup X_1$. Then player II represents Y as a union $Y = Y_0 \cup Y_1$. Now player I chooses whether the game continues from the position (X_0, Y_0) or from the position (X_1, Y_1).*

Duplication move: *Player I decides that the game should continue from the new position*

$$(X(M/x_{i_m}), Y(N/x_{i_m})).$$

Supplementing move: *Player I chooses a function $F : X \to M$. Then player II chooses a function $G : Y \to N$. Then the game continues from the position $(X(F/x_{i_m}), Y(G/x_{i_m}))$.*

After n moves the position (X_n, Y_n) is reached and the game ends. Player II is the winner if

$$\mathcal{M} \models_{X_n} \phi \Rightarrow \mathcal{N} \models_{Y_n} \phi$$

holds for all atomic and negated atomic and dependence formulas of the form $\phi(x_0, ..., x_{i_n-1})$. Otherwise player I wins.

This is a game of perfect information and the concept of winning strategy is defined as usual. By the Gale-Stewart theorem the game is determined.

Definition 83 *1.* $\mathrm{qr}(\phi) = 0$ *if ϕ is atomic or a dependence formula.*

2. $\mathrm{qr}(\phi \vee \psi) = \max(\mathrm{qr}(\phi), \mathrm{qr}(\psi)) + 1$.

3. $\mathrm{qr}(\exists x_n \phi) = \mathrm{qr}(\phi) + 1$.

4. $\mathrm{qr}(\neg\phi)$:

(a) $\mathrm{qr}(\neg\phi) = 0$ *if* ϕ *is atomic or a dependence formula.*

(b) $\mathrm{qr}(\neg\neg\phi) = \mathrm{qr}(\phi)$.

(c) $\mathrm{qr}(\neg(\phi \vee \psi)) = \max(\mathrm{qr}(\neg\phi), \mathrm{qr}(\neg\psi))$.

(d) $\mathrm{qr}(\neg\exists x_n\phi) = \mathrm{qr}(\neg\phi) + 1$.

Let Fml_n^m *be the set of formulas* ϕ *of* $\mathcal{D}$ *with* $\mathrm{qr}\,\phi \leq m$ *and with free variables among* $x_0, ..., x_{n-1}$. *We write* $\mathcal{M} \Rrightarrow_{\mathcal{D}}^n \mathcal{N}$ *if* $\mathcal{M} \models \phi$ *implies* $\mathcal{N} \models \phi$ *for all* ϕ *in* Fml_0^n, *and* $\mathcal{M} \equiv_{\mathcal{D}}^n \mathcal{N}$ *if* $\mathcal{M} \models \phi$ *is equivalent to* $\mathcal{N} \models \phi$ *for all* ϕ *in* Fml_0^n.

Note that there are for each n and m, up to logical equivalence, only finitely many formulas in Fml_n^m.

Theorem 84 *Suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are models of the same vocabulary. Then the following are equivalent:*

(1) *Player II has a winning strategy in the game* $\mathrm{EF}_n(\mathcal{M}, \mathcal{N})$.

(2) $\mathcal{M} \Rrightarrow_{\mathcal{D}}^n \mathcal{N}$.

Proof. We prove the equivalence, for all n, of the following two statements:

$(3)_m$ Player II has a winning strategy in the game $\mathrm{EF}_m(\mathcal{M}, \mathcal{N})$ in position (X, Y), where $X \subseteq M^n$ and $Y \subseteq N^n$.

$(4)_m$ If ϕ is a formula in Fml_n^m, then

$$\mathcal{M} \models_X \phi \Rightarrow \mathcal{N} \models_Y \phi. \tag{4.8}$$

The proof is by induction on m. For each m we prove the claim simultaneously for all n. The case $m = 0$ is true by construction. Let us then assume $(3)_m \iff (4)_m$ as an induction hypothesis. Assume now $(3)_{m+1}$ and let ϕ be a formula in Fml_n^{m+1} such that $\mathcal{M} \models_X \phi$. As part of the induction hypothesis we assume that the claim (4.8) holds for formulas shorter than ϕ.

Case 1: $\phi = \psi_0 \vee \psi_1$, where $\psi_0, \psi_1 \in \mathrm{Fml}^m_n$. Since $\mathcal{M} \models_X \phi$, there are X_0 and X_1 such that $X = X_0 \cup X_1$, $\mathcal{M} \models_{X_0} \psi_0$ and $\mathcal{M} \models_{X_1} \psi_1$. We let I play $\{X_0, X_1\}$. Then II plays according to her winning strategy $\{Y_0, Y_1\}$. Since the next position in the game can be either one of (X_0, Y_0), (X_1, Y_1), we can apply the induction hypothesis to both. This yields $\mathcal{N} \models_{Y_0} \psi_0$ and $\mathcal{N} \models_{Y_1} \psi_1$. Thus $\mathcal{N} \models_Y \phi$.

Case 2: $\phi = \exists x_n \psi$, where $\psi \in \mathrm{Fml}^m_{n-1}$. Since $\mathcal{M} \models_X \phi$, there is a function $F : X \to M$ such that $\mathcal{M} \models_{X(F/x_n)} \psi$. We let I play F. Then II plays according to her winning strategy a function $G : Y \to N$ and the game continues in position $(X(F/x_n), Y(G/x_n))$. The induction hypothesis gives $\mathcal{N} \models_{Y(G/x_n)} \psi$. Now $\mathcal{N} \models_Y \phi$ follows.

Case 3: $\phi = \neg\neg\psi$, $n = \max(n_0, n_1)$. Since $\mathcal{M} \models_X \phi$, we have $\mathcal{M} \models_X \psi$. By the induction hypothesis, $\mathcal{N} \models_Y \psi$. Thus $\mathcal{N} \models_Y \phi$.

Case 4: $\phi = \neg(\psi_0 \vee \psi_1)$, where $\psi_0, \psi_1 \in \mathrm{Fml}^m_n$. Since $\mathcal{M} \models_X \phi$, we have $\mathcal{M} \models_X \neg\psi_0$ and $\mathcal{M} \models_X \neg\psi_1$. By the induction hypothesis, $\mathcal{N} \models_Y \neg\psi_0$ and $\mathcal{N} \models_Y \neg\psi_1$. Thus $\mathcal{N} \models_Y \phi$.

Case 5: $\phi = \neg\exists x_n \psi$, where $\neg\psi \in \mathrm{Fml}^m_{n+1}$. By assumption, $\mathcal{M} \models_{X(M/x_n)} \neg\psi$. We let now I demand that the game continues in the duplicated position $(X(M/x_n), Y(N/x_n))$. The induction hypothesis gives $\mathcal{N} \models_{Y(N/x_n)} \neg\psi$. Now $\mathcal{N} \models_Y \phi$ follows trivially.

To prove the converse implication, assume $(4)_{m+1}$. To prove $(3)_{m+1}$ we consider the possible moves that player I can make in the position (X, Y).

Case i: Player **I** writes $X = X_0 \cup X_1$. Let ϕ_j, $j < k$, be a complete list (up to logical equivalence) of formulas in Fml^m_n. Since $\mathcal{M} \models_{X_0} \neg \bigvee_{\mathcal{M} \models_{X_0} \phi_j} \neg\phi_j$ and $\mathcal{M} \models_{X_1} \neg \bigvee_{\mathcal{M} \models_{X_1} \phi_j} \neg\phi_j$, we have

$$\mathcal{M} \models_X (\neg \bigvee_{\mathcal{M} \models_{X_0} \phi_j} \neg\phi_j) \vee (\neg \bigvee_{\mathcal{M} \models_{X_1} \phi_j} \neg\phi_j).$$

Note that $\mathrm{qr}(\neg\bigvee_{\mathcal{M}\models_{X_1}\phi_j}\neg\phi_j) = \max_{\mathcal{M}\models_{X_1}\phi_j}(\neg\neg\phi_j) = \max_{\mathcal{M}\models_{X_1}\phi_j}\phi_j \leq m$. Therefore by $(4)_{m+1}$

$$\mathcal{N} \models_Y (\neg \bigvee_{\mathcal{M}\models_{X_0}\phi_j} \neg\phi_j) \vee (\neg \bigvee_{\mathcal{M}\models_{X_1}\phi_j} \neg\phi_j).$$

Thus $Y = Y_0 \cup Y_1$ such that $\mathcal{N} \models_{Y_0} \neg\bigvee_{\mathcal{M}\models_{X_0}\phi_j}\neg\phi_j$ and $\mathcal{N} \models_{Y_1} \neg\bigvee_{\mathcal{M}\models_{X_1}\phi_j}\neg\phi_j$. By this and the induction hypothesis, player **II** has a winning strategy in the positions $(X_0, Y_0), (X_1, Y_1)$. Thus she can play $\{Y_0, Y_1\}$ and maintain her winning strategy.

Case ii: Player **I** decides that the game should continue from the new position $(X(M/x_n), Y(m/x_n))$. We claim that

$$\mathcal{M} \models_{X(M/x_n)} \phi \Rightarrow \mathcal{N} \models_{Y(N/x_n)} \phi$$

for all $\phi \in \mathrm{Fml}^m_{n+1}$. From this the induction hypothesis would imply that II has a winning strategy in the position $(X(M/x_n), Y(N/x_n))$. So let us assume $\mathcal{M} \models_{X(M/x_n)} \phi$, where $\phi \in \mathrm{Fml}^m_{n+1}$. By definition,

$$\mathcal{M} \models_X \neg\exists x_n \neg\phi.$$

Since $\neg\exists x_n\neg\phi \in \mathrm{Fml}^{m+1}_n$, $(4)_{m+1}$ gives $\mathcal{N} \models_Y \neg\exists x_n\neg\phi$ and $\mathcal{N} \models_{Y(N/x_n)} \phi$ follows.

Case iii: Player I chooses a function $F : X \to M$. Let $\phi_i, i < M$ be a complete list (up to logical equivalence) of formulas in Fml^m_{n+1}. Now $\mathcal{M} \models_X \exists x_n \neg\bigvee_{\mathcal{M}\models_{X(F/x_n)}\phi_i}\neg\phi_i$. Note that

$$\begin{aligned}
\mathrm{qr}(\exists x_n \neg \bigvee_{\mathcal{M}\models_{X(F/x_n)}\phi_j} \neg\phi_j) &= \mathrm{qr}(\neg \bigvee_{\mathcal{M}\models_{X(F/x_n)}\phi_j} \neg\phi_j) + 1 \\
&= (\max_{\mathcal{M}\models_{X(F/x_n)}\phi_j} \mathrm{qr}(\neg\neg\phi_j)) + 1 \\
&= (\max_{\mathcal{M}\models_{X(F/x_n)}\phi_j} \mathrm{qr}(\phi_j)) + 1 \leq m+1,
\end{aligned}$$

and hence by $(4)_{m+1}$, $\mathcal{N} \models_Y \exists x_n \neg \bigvee_{\mathcal{M}\models_{X(F/x_n)}\phi_i} \neg\phi_i$. Thus there is a function $G : Y \to N$ such that

$$\mathcal{N} \models_{Y(G/x_n)} \neg \bigvee_{\mathcal{M}\models_{X(F/x_n)}\phi_i} \neg\phi_i.$$

The game continues from position $(X(F/x_n), Y(G/x_n)$. Given that now $\mathcal{M} \models_{X(F/x_n)} \phi \Rightarrow \mathcal{N} \models_{Y(G/x_n)} \phi$ for all $\phi \in \mathrm{Fml}^m_{n+1}$, the induction hypothesis implies that II has a winning strategy in position $(X(F/x_n), Y(G/x_n))$. $\Box$

Corollary 85 *Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of the same vocabulary. Then the following are equivalent:*

(1) $\mathcal{M} \Rrightarrow_{\mathcal{D}} \mathcal{N}$.

(2) *For all natural numbers n, player II has a winning strategy in the game* $\mathrm{EF}_n(\mathcal{M}, \mathcal{N})$.

Corollary 86 *Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of the same vocabulary. Then the following are equivalent:*

(1) $\mathcal{M} \equiv_{\mathcal{D}} \mathcal{N}$.

(2) *For all natural numbers n, player II has a winning strategy both in the game* $\mathrm{EF}_n(\mathcal{M}, \mathcal{N})$ *and in the game* $\mathrm{EF}_n(\mathcal{N}, \mathcal{M})$.

The two games $\mathrm{EF}_n(\mathcal{M}, \mathcal{N})$ and $\mathrm{EF}_n(\mathcal{N}, \mathcal{M})$ can be put together into one game by simply making the moves of the former symmetric with respect to $\mathcal{M}$ and $\mathcal{N}$. Then player II has a winning strategy in this new game if and only if $\mathcal{M} \equiv^n_{\mathcal{D}} \mathcal{N}$. Instead of a game we could have used a notion of a back-and-forth sequence.

The Ehrenfeucht-Fraïssé game can be used to prove non-expressibility results for $\mathcal{D}$, but we do not yet have examples where a more direct proof using compactness, interpolation and Löwenheim-Skolem theorems would not be simpler.

Proposition 87 *There are countable models $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \Rrightarrow_{\mathcal{D}} \mathcal{N}$, but $\mathcal{N} \not\Rrightarrow_{\mathcal{D}} \mathcal{M}$.*

Proof. Let $\mathcal{M}$ be the standard model of arithmetic. Let $\Phi_n, n \in \omega$ be the list of all Σ^1_1-sentences true in $\mathcal{M}$. Suppose $\Phi_n = \exists R^n_1 ... \exists R^n_{k_n} \phi_n$. Let $\mathcal{M}^*$ be an expansion of $\mathcal{M}$ in which each ϕ_n is true. Let $\mathcal{N}^*$ be a countable non-standard elementary extension of $\mathcal{M}^*$. Let $\mathcal{N}$ be the reduct of $\mathcal{N}^*$ to the language of arithmetic. By construction, $\mathcal{M} \Rrightarrow_{\mathcal{D}} \mathcal{N}$. On the other hand, $\mathcal{N} \not\Rrightarrow_{\mathcal{D}} \mathcal{M}$ as non-wellfoundedness of the integers in $\mathcal{N}$ can be expressed by a sentence of $\mathcal{D}$. $\Box$

Proposition 88 *Suppose K is a model class*[3] *and n is a natural number. Then the following are equivalent:*

(1) *K is definable in dependence logic by a sentence in* Fml^n_0.

(2) *K is closed under the relation $\Rrightarrow^n_{\mathcal{D}}$.*

Proof. Suppose K is the class of models of $\phi \in \mathrm{Fml}^n_0$. If $\mathcal{M} \models \phi$ and $\mathcal{M} \Rrightarrow^n_{\mathcal{D}} \mathcal{N}$, then by definition, $\mathcal{N} \models \phi$. Conversely, suppose K is closed under $\Rrightarrow^n_{\mathcal{D}}$. Let $\phi_{\mathcal{M}} = \neg \bigvee \{\neg\phi : \phi \in \mathrm{Fml}^n_0, \mathcal{M} \models \phi\}$, where the conjunction is taken over a finite set which covers all such ϕ up to logical equivalence. Let θ be the disjunction of all $\phi_{\mathcal{M}}$, where $\mathcal{M} \in K$. Again we take the disjunction over a finite set up to logical equivalence. We show that K is the class of models of θ. If $\mathcal{M} \in K$ then $\mathcal{M} \models \phi_{\mathcal{M}}$, whence $\mathcal{M} \models \theta$. On the other hand suppose $\mathcal{M} \models \phi_{\mathcal{N}}$ for some $\mathcal{N} \in K$. Now $\mathcal{N} \Rrightarrow^n_{\mathcal{D}} \mathcal{M}$, for if $\mathcal{N} \models \phi$ and $\phi \in \mathrm{Fml}^n_{\emptyset}$, then ϕ is logically equivalent with one of the conjuncts of $\phi_{\mathcal{N}}$, whence $\mathcal{M} \models \phi$. As K is closed under $\Rrightarrow^n_{\mathcal{D}}$, we have $\mathcal{M} \in K$. $\Box$

Corollary 89 *Suppose K is a model class. Then the following are equivalent:*

(1) *K is definable in dependence logic.*

[3] A model class is a class of models, closed under isomorphism, of the same vocabulary.

(2) *There is a natural number n such that K is closed under the relation $\Rrightarrow^n_{\mathcal{D}}$.*

The above corollary also gives a characterization of Σ^1_1-definability in second order logic. No assumptions about cardinalities are involved, so if we restrict ourselves to finite models we get a characterization of NP-definability.

Exercise 99 *Suppose $L = \emptyset$ and $\mathcal{M}$ and $\mathcal{N}$ are L-structures such that $M = \{0, 1, 2\}$ and $N = \{a, b, c, d\}$, where a, b, c and d are distinct. The game $\mathrm{EF}_3(\mathcal{M}, \mathcal{N})$ is in the position:*

$$X: \begin{array}{|c|c|c|} x_0 & x_1 & x_2 \\ \hline 0 & 2 & 2 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \\ \hline \end{array}, \quad Y: \begin{array}{|c|c|c|} x_0 & x_1 & x_2 \\ \hline a & c & b \\ b & b & c \\ c & a & a \\ d & d & d \\ \hline \end{array}$$

Who won the game?

Exercise 100 *Suppose $L = \emptyset$ and $\mathcal{M}$ and $\mathcal{N}$ are L-structures such that $M = \{0, 1, 2, 3\}$ and $N = \{a, b, c, d\}$, where a, b, c and d are distinct. The game $\mathrm{EF}_3(\mathcal{M}, \mathcal{N})$ is in the position:*

$$X: \begin{array}{|c|c|} x_0 & x_1 \\ \hline 0 & 2 \\ 1 & 3 \\ 2 & 3 \\ 3 & 2 \\ \hline \end{array}, \quad Y: \begin{array}{|c|c|} x_0 & x_1 \\ \hline a & b \\ b & c \\ c & d \\ d & a \\ \hline \end{array}$$

Can you spot a good splitting move for player **I**?

Exercise 101 *Suppose $L = \{P\}$ and $\mathcal{M}$ and $\mathcal{N}$ are L-structures such that $M = \{0, 1, 2\}$, $P^{\mathcal{M}} = \{0\}$, $N = \{a, b, c\}$, and $P^{\mathcal{N}} = \emptyset$, where a, b, and c are distinct. The game $\mathrm{EF}_3(\mathcal{M}, \mathcal{N})$ is in the position:*

$$X: \begin{array}{|c|c|} x_0 & x_1 \\ \hline 0 & 2 \\ 1 & 1 \\ 2 & 1 \\ \hline \end{array}, \quad Y: \begin{array}{|c|c|} x_0 & x_1 \\ \hline a & b \\ b & b \\ c & a \\ \hline \end{array}$$

Can you spot a good supplementing move for player **I**?

Exercise 102 *Let $L = \emptyset$. Show that there is no sentence ϕ of vocabulary L in Fml_0^2 such that $\mathcal{M} \models \phi$ if and only if M is infinite.*

Exercise 103 *Let $L = \emptyset$. Show that there is no sentence ϕ of vocabulary L in Fml_0^2 such that for finite models $\mathcal{M}$ we have: $\mathcal{M} \models \phi$ if and only if $|M|$ is even.*

Exercise 104 *Show that there is a countable model which is not $\mathcal{D}$-equivalent to any uncountable models.*

Exercise 105 *Show that $(\mathbb{R}, \mathbb{N}) \Rrightarrow_{\mathcal{D}} (\mathbb{Q}, \mathbb{N})$.*

Exercise 106 *Show that $(\omega + \omega^* + \omega, <) \not\Rrightarrow_{\mathcal{D}} (\omega, <)$.*

Exercise 107 *Show that if $\mathcal{M} \Rrightarrow_{\mathcal{D}} \mathcal{N}$ and $\mathcal{N}$ is a connected graph, then so is $\mathcal{M}$.*

Exercise 108 *Give three models $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$ such that $\mathcal{M} \Rrightarrow_{\mathcal{D}} \mathcal{M}'$ and $\mathcal{M} \Rrightarrow_{\mathcal{D}} \mathcal{M}''$, but $\mathcal{M}' \not\Rrightarrow_{\mathcal{D}} \mathcal{M}''$ and $\mathcal{M}'' \not\Rrightarrow_{\mathcal{D}} \mathcal{M}'$.*

Exercise 109 *Give models $\mathcal{M}_n$ such that for all $n \in \mathbb{N}$ we have $\mathcal{M}_{n+1} \Rrightarrow_{\mathcal{D}} \mathcal{M}_n$, but $\mathcal{M}_n \not\Rrightarrow_{\mathcal{D}} \mathcal{M}_{n+1}$.*

Chapter 5

Complexity

5.1 Decision and Other Problems

The basic problem this chapter considers is how difficult is it to decide some basic questions concerning the relation $\mathcal{M} \models \phi$, when $\mathcal{M}$ is a structure and ϕ is a $\mathcal{D}$-sentence? Particular questions we study are:

Decision Problem: Is ϕ valid, i.e. does $\mathcal{M} \models \phi$ hold for *all* $\mathcal{M}$.

Nonvalidity Problem: Is ϕ *nonvalid*, i.e. does ϕ *avoid* some model, i.e. does $\mathcal{M} \models \phi$ fail for some model $\mathcal{M}$.

Consistency Problem: Is ϕ *consistent*, i.e. does ϕ have a model, i.e. does $\mathcal{M} \models \phi$ hold for *some* $\mathcal{M}$.

Inconsistency Problem: Is ϕ *inconsistent*, i.e. does ϕ *avoid* all models, i.e. is $\mathcal{M} \models \phi$ true for no model $\mathcal{M}$ at all.

Obviously such questions depend on the vocabulary. In a unary vocabulary it may be easier to decide some of the above questions. On the other hand, if at least one binary predicate is allowed, then the questions are as hard as for any other vocabulary, as there are coding techniques that allow us to code bigger vocabularies into one binary predicate. We can ask the same questions about models of particular theories, like groups, linear orders, fields, graphs, and so on. Furthermore, we can ask these questions in the framework of finite models.

By definition, ϕ is nonvalid if and only if ϕ is not valid, and ϕ is inconsistent
So it suffices to concentrate on the Decision Problem and the Consis-

tency Problem. In first order logic we have the further equivalence ϕ is consistent if and only if $\neg\phi$ is not valid. So if we crack the decision problem for first order logic everything else follows. Indeed, the Gödel Completeness Theorem tells us that a first order sentence is valid if and only it is provable. Hence the Decision Problem of first order logic is Σ^0_1 (i.e. recursively enumerable)

The Consistency Problem for dependence logic can be reduced to that of first order logic by the equivalence ϕ is consistent if and only if $\tau_{1,\phi}$ is consisten

5.2 Some Set Theory

The complexity of the Decision Problem and of the Nonvalidity Problem of dependence logic is so great that we have to move from complexity measures on the integers to complexity measures in set theory. With this in mind we recall some elementary concepts from set theory:

A set a is *transitive* if $c \in b$ and $b \in a$ imply $c \in a$ for all a and b. The *transitive closure* $TC(a)$ of a set a is the intersection of all transitive supersets of a, or in other words $a \cup (\cup a) \cup (\cup \cup a) \ldots$. Intuitively, $TC(a)$ consists of elements of a, elements of elements of a, elements of elements of elements of a, etc. Define $H_\kappa = \{a : |TC(a)| < \kappa\}$.

A priori it is not evident that H_κ is a set. However, this can be easily proved with another useful concept from set theory, namely the concept of rank. The *rank* $\mathrm{rk}(a)$ of a set a is defined recursively as follows: $\mathrm{rk}(a) = \sup\{\mathrm{rk}(b) + 1 : b \in a\}$. Recall the definition of the cumulative hierarchy in (2.1). Now $V_\alpha = \{x : \mathrm{rk}(x) < \alpha\}$ and we can prove that H_κ is indeed a set:

Lemma 90 *For all infinite cardinals κ we have $H_\kappa \subseteq V_\kappa$.*

Proof. Suppose $x \in H_\kappa$, i.e. $|TC(x)| < \kappa$. We claim that $\mathrm{rk}(x) < \kappa$, i.e. $|\mathrm{rk}(x)| < \kappa$. It suffices to show that $|\mathrm{rk}(x)| \leq |TC(x)|$. This follows by the Axiom of Choice, if we show that there is, for all x, an onto function from $TC(x)$ onto $\mathrm{rk}(x)$. This function is in fact the function $z \mapsto \mathrm{rk}(z)$. Suppose the claim that $\mathrm{rk} \lceil TC(y)$ maps $TC(y)$ onto $\mathrm{rk}(y)$ holds for all $y \in x$. We show that it holds for x. To this end, suppose $\alpha < \mathrm{rk}(x)$. By definition, $\alpha \leq \mathrm{rk}(y)$ for some $y \in x$. If $\alpha < \mathrm{rk}(y)$, then by the induction hypothesis, there is $z \in TC(y)$ such that $\mathrm{rk}(z) = \alpha$. Now $z \in TC(x)$, so we are done. The other case is that $\alpha = \mathrm{rk}(y)$. Since $y \in TC(x)$ we are done again. $\square$

The converse of Lemma 90 is certainly not true in general. For example, the set $V_{\aleph_1}$ has sets such as $\mathcal{P}(\omega)$ which cannot be in $H_{\aleph_1}$. However, let us define the *Beth numbers* as follows: $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and $\beth_\nu = \lim_{\alpha<\nu} \beth_\alpha$ for limit ν. For any κ there is $\lambda \geq \kappa$ such that $\lambda = \beth_\lambda$, as can be seen as follows: let $\kappa_0 = \kappa$, $\kappa_{n+1} = \beth_{\kappa_n}$ and $\lambda = \lim_{n<\omega} \kappa_n$. Then $\lambda = \beth_\lambda$. It is easy to see by induction on α that $|V_{\omega+\alpha}| = \beth_\alpha$.

Lemma 91 *If $\kappa = \beth_\kappa$, then $H_\kappa = V_\kappa$.*

Proof. The claim $H_\kappa \subseteq V_\kappa$ follows from Lemma 90. On the other hand, if $x \in V_\kappa$, say $x \in V_\alpha$, where $\alpha = \omega + \alpha < \kappa$, then $|TC(x)| \leq |V_\alpha| = \beth_\alpha < \beth_\kappa = \kappa$, so $x \in H_\kappa$. $\square$

We now recall an important hierarchy in set theory:

Definition 92 ([20]) *The* Levy Hierarchy *of formulas of set theory is obtained as follows: The Σ_0-formulas, which are at the same time called Π_0-formulas, are all formulas in the vocabulary $\{\in\}$ obtained from atomic formulas by the operations $\neg, \vee, \wedge$ and the* bounded quantifiers*:$\exists x_0(x_0 \in x_1 \wedge \phi)$ and $\forall x_0(x_0 \in x_1 \rightarrow \phi)$. The Σ_{n+1}-formulas are obtained from Π_n-formulas by existential quantification. The Π_{n+1}-formulas are obtained from Σ_n-formulas by existential quantification.*

A basic property of the Σ_1-formulas is captured by the following lemma:

Lemma 93 *For any uncountable cardinal κ, $a_1, ..., a_n \in H_\kappa$, and Σ_1-formula $\phi(x_1, ..., x_n)$: $(H_\kappa, \epsilon) \models \phi(a_1, ..., a_n)$, if and only if $\phi(a_1, ..., a_n)$.*

Proof. The "only if" part is very easy (see Exercise 110). For the more difficult direction suppose $\phi(x_1, ..., x_n)$ is of the form $\exists x_0 \psi(x_0, x_1, ..., x_n)$, where $\psi(x_0, x_1, ..., x_n)$ is Σ_0, and there is an a_0 such that $\phi(a_0, a_1, ..., a_n)$. Let α be large enough that $a_0, ...a_n \in V_\alpha$. Then $(V_\alpha, \epsilon) \models \psi(a_0, ..., a_n)$ by Exercise 110. Thus $(V_\alpha, \epsilon) \models \phi(a_1, ..., a_n)$. Let $\mathcal{M}$ be an elementary submodel of the model (V_α, ϵ) such that $TC(\{a_1, ..., a_n\}) \subseteq M$ and $|M| < \kappa$. By Mostowski's Collapsing Lemma (see Exercise 111) there is a transitive model (N, ϵ) and an isomorphism $\pi : (N, \epsilon) \cong \mathcal{M}$ such that $a_1, ..., a_n \in N$ and $\pi(a_i) = a_i$ for each i. Thus $(N, \epsilon) \models \phi(a_1, ..., a_n)$. Since $x \in N$ implies $TC(x) \subseteq N$, and $|N| < \kappa$, we have $N \subseteq H_\kappa$, and hence again by absoluteness, $(H_\kappa, \epsilon) \models \phi(a_1, ..., a_n)$. $\Box$

An intuitive picture of a Σ_1 statement $\exists x_0 \phi(a, x_0)$, where ϕ is Σ_0, is that we search through the universe for an element b such that the relatively simple statement $\phi(a, b)$ becomes true. By Lemma 93 we need only look near where $a_1, ..., a_n$ are. This means that satisfying a Σ_1-sentence is from a set-theoretic point of view not very complex, although it may still be at least as difficult as checking whether a recursive binary relation on $\mathbb{N}$ is well-founded (which is a Π^1_1-complete problem).

In contrast, to check whether a Σ_2 sentence $\exists x_0 \forall x_1 \phi(a, x_0, x_1)$ is true one has to search through the whole universe for a b such that $\forall x_1 \phi(a, b, x_1)$ is true. Now we cannot limit ourselves to search close to a as we may have to look close to b, too. So in the end we have to go through the whole universe in search of b. This means that checking the truth of a Σ_2-sentence is of extremely high complexity. Any of the following statements can be expressed as the truth of a Σ_2-sentence:

1. The Continuum Hypothesis, i.e. $2^{\aleph_0} = \aleph_1$
2. The failure of the Continuum Hypothesis, i.e. $2^{\aleph_0} \neq \aleph_1$
3. $V \neq L$
4. There is an inaccessible cardinal

5. There is a measurable cardinal

We now have the notions at hand for filling in the complexity table for dependence logic, even if we have not proved anything yet:

Dependence logic	
Problem	Complexity
Decision problem	Π_2
Nonvalidity problem	Σ_2
Consistency problem	Π^0_1
Inconsistency problem	Σ^0_1

The proof is given in the next section.

Exercise 110 *[Absoluteness of Σ_0] For any transitive M, $a_1, ..., a_n \in M$, and Σ_0-formula $\phi(x_1, ..., x_n)$: $(M, \epsilon) \models \phi(a_1, ..., a_n)$, if and only if $\phi(a_1, ..., a_n)$.*

Exercise 111 *[Mostowski's Collapsing Lemma] Suppose (M, E) is a well-founded model of the Axiom of Extensionality: $\forall x_0 \forall x_1 (\forall x_2 (x_2 \in x_0 \leftrightarrow x_2 \in x_1) \rightarrow x_0 = x_1)$. Show that the equation $\pi(x) = \{\pi(y) : yEx\}$ defines an isomorphism between (M, E) and $(N, \in)$, where N is a transitive set. Show also that $\pi(x) = x$ for every $x \in M$ which is E-transitive (i.e. zEy and yEx imply zEx).*

5.3 Σ_2-completeness in set theory

Let us now go a little deeper into details. We identify problems with sets $X \subseteq \mathbb{N}$. The problem X in such a case is really the problem of deciding whether a given n is in X or not. A problem $P \subseteq \mathbb{N}$ is called Σ_n-definable if there is a Σ_n-formula $\phi(x_0)$ of set theory such that $n \in P \iff \phi(n)$. A problem $P \subseteq \mathbb{N}$ is called Σ_n-complete if the problem itself is Σ_n-definable and, moreover, for every Σ_n-definable set

$X \subseteq \mathbb{N}$ there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $n \in X \iff f(n) \in P$. The concepts of a Π_n-definable set and a Π_n-complete set are defined analogously.

Theorem 94 *The Decision Problem of dependence logic is* Π_2*-complete in set theory.*

Proof. Without loss of generality, we consider the decision problem in the context of a vocabulary consisting of just one binary predicate. Let us first observe that the predicate $x = \mathcal{P}(y)$ is Π_1-definable:

$$x = \mathcal{P}(y) \iff \forall z(z \in x \leftrightarrow \forall u \in z(u \in y)).$$

Let $On(x)$ be the Σ_0-predicate "x is an ordinal," i.e. x is a transitive set of transitive sets. We can Π_1-define the property $R(x)$ of x of being equal to some V_α, where α is a limit ordinal. Let $\mathrm{Str}(x)$ be the first-order formula in the language of set theory which says that x is a structure of the vocabulary containing just one binary predicate symbol. If ϕ is a sentence of $\mathcal{D}$, let $\mathrm{Sat}_\phi(x)$ be the first-order formula in the language of set theory which says "$\mathrm{Str}(x)$ and ϕ is true in the structure x." Let $\mathrm{Relsat}_\phi(x, y)$ be the first-order formula in the language of set theory which says "$x \in y$ and if $\mathrm{Str}(x)$, then $\mathrm{Sat}_\phi(x)$ is true when relativized to the set y." Thus $\models \phi$ if and only if $\forall x(\mathrm{Str}(x) \to \mathrm{Sat}_\phi(x))$. Note that for limit α and $a \in V_\alpha$:

$$\mathrm{Sat}_\phi(a) \iff (V_\alpha \models \mathrm{Sat}_\phi(a)).$$

Thus a sentence ϕ of $\mathcal{D}$ is valid if and only if

$$\forall x(R(x) \to \forall y \in x \mathrm{Relsat}_\phi(y, x)).$$

We have proved that the decision problem of $\mathcal{D}$ is Π_2-definable. Suppose then A is an arbitrary Π_2-definable set of integers. Let $\forall x \exists y \psi(n, x, y)$ be the Π_2-definition. Let ϕ_n be the first-order sentence $\forall x \exists y \psi(n, x, y)$, where n is a defined term. We claim

$$n \in A \iff \models \theta \vee \phi_n,$$

where θ is a sentence of $\mathcal{D}$ which is true in every model except the models $(V_\kappa, \in)$, $\kappa = \beth_\kappa$. Suppose first $n \in A$, i.e. $\forall x \exists y \psi(n, x, y)$. Suppose $(V_\kappa, \in)$ is a given model in which θ is not true. We prove $V_\kappa \models \forall x \exists y \psi(n, x, y)$. Suppose $a \in V_\kappa$. By the above Lemmas, there is $b \in V_\kappa$ such that $\psi(n, a, b)$. We have proved $\models \theta \vee \phi_n$. Conversely, suppose $\models \theta \vee \phi_n$. To prove $\forall x \exists y \psi(n, x, y)$, let a be given. Let κ be an infinite cardinal such that $\kappa = \beth_\kappa$ and $a \in H_\kappa$. Then there is $b \in H_\kappa$ with $H_\kappa \models \psi(n, a, b)$. Now $\psi(n, a, b)$ follows. $\square$

Corollary 95 *The decision problem of dependence logic is not Σ_2-definable in set theory.*

The fact that the decision problem of dependence logic is not Σ_n^m for any $m, n < \omega$ follows easily from this. Moreover, it follows that we cannot in general express "ϕ is valid," for $\phi \in \mathcal{D}$, even by searching through the whole set-theoretical universe for a set x such that a universal quantification over the subsets of x would guarantee the validity of ϕ. In contrast, to check validity of a first-order sentence, one needs only search through all natural numbers and then perform a finite polynomial calculation on that number.

Exercise 112 *Give a Σ_2-definition of the property of x being equal to some V_κ, where $\kappa = \beth_\kappa$.*

Exercise 113 *Show that for limit α and $a \in V_\alpha$:* $\mathrm{Sat}_\phi(a) \iff (V_\alpha \models \mathrm{Sat}_\phi(a))$.

Exercise 114 *Give a sentence of $\mathcal{D}$ which is true in every model except (mod $\cong$) the models $(V_\kappa, \in)$, $\kappa = \beth_\kappa$.*

Exercise 115 *Use Theorem 94 to prove that the decision problem of $\mathcal{D}$ is not arithmetical, i.e. not first order definable definable in $(\mathbb{N}, +, \cdot, <, 0, 1)$. (In fact the same proof shows that it is not Σ_n^m-definable for any $m, n < \omega$).*

Exercise 116 *Show that the problem: Does* $\mathcal{M} \models \phi$ *hold for all countable* $\mathcal{M}$*? is not arithmetical, and the problem: Does* $\mathcal{M} \models \phi$ *hold for some countable* $\mathcal{M}$*? is a* Π^0_1 *property of* $\phi \in \mathcal{D}$*.*

Exercise 117 *Show that there is a sentence* ϕ *of* $\mathcal{D}$ *such that* ϕ *avoids some model if and only if there is an inaccessible cardinal.*

Chapter 6

Team Logic

The negation $\neg$ of dependence logic does not satisfy the Law of Excluded Middle and is therefore not the classical Boolean negation. This is clearly manifested by the existence of non-determined sentences ϕ in $\mathcal{D}$. In such cases the failure of $\mathcal{M} \models \phi$ does not imply $\mathcal{M} \models \neg\phi$. Hintikka [10] introduced *extended independence friendly logic* by taking the Boolean closure of his independence friendly logic. We take the further action of making classical negation $\sim$ one of the logical operations on a par with other propositional operations and quantifiers. This yields an extension TL of the Boolean closure of $\mathcal{D}$. We call the new logic *team logic.*

The basic concept of both team logic TL and dependence logic $\mathcal{D}$ is the concept of dependence $=(t_1, ..., t_n)$. In very simple terms, what happens is that, while we can say "x_1 **is** a function of x_0" with $=(x_0, x_1)$ in $\mathcal{D}$, we will be able to say "x_1 is **not** a function of x_0" with $\sim =(x_0, x_1)$ in team logic.

While we define team logic we have to restrict $\neg$ to atomic formulas. The game-theoretic intuition behind $\neg\phi$ is that it says something about "the other player." The introduction of $\sim$ unfortunately ruins the basic game-theoretic intuition, and there is no "other player" anymore. If ϕ is in $\mathcal{D}$, then $\sim\phi$ has the meaning "**II** does not have a winning strategy," but it is not clear what the meaning of $\neg\sim\phi$ would be. We also change notation by using $\phi \otimes \psi$ ("tensor") for what was $\phi \vee \psi$ in $\mathcal{D}$. The reason for this is that by means of $\sim$ and $\wedge$ we can actually define the classical Boolean disjunction $\phi \vee \psi$, which really says the team is of type

ϕ or of type ψ. Likewise, we adopt the notation $!x_n\phi$ ("shriek") for the quantifier that in $\mathcal{D}$ was denoted by $\forall x_n\phi$. For motivation of the new symbols, see Remark 99.

6.1 Formulas of Team Logic

In this section we give the syntax and semantics of team logic and indicate some basic principles.

Definition 96 *Suppose L is a vocabulary. The formulas of* team logic TL *are:*

Atomic	*Name*	*Complex*	*Name*
$t_1 = t_n$	*equation*	$\phi \otimes \psi$	*tensor*
$\neg t_1 = t_n$	*dual equation*	$\phi \wedge \psi$	*conjunction*
$Rt_1...t_n$	*relation*	$\sim\phi$	*negation*
$\neg Rt_1...t_n$	*dual relation*	$\exists x_n\phi$	*existential*
$=(t_1, ..., t_n)$	*dependence*	$!x_n\phi$	*shriek*
$\neg=(t_1, ..., t_n)$	*dual dependence*		

Definition 97 *The semantics of team logic is defined as follows:*

TL1 $\mathcal{M} \models_X t_1 = t_2$ *iff for all* $s \in X$ *we have* $t_1^{\mathcal{M}}\langle s\rangle = t_2^{\mathcal{M}}\langle s\rangle$.

TL2 $\mathcal{M} \models_X \neg t_1 = t_2$ *iff for all* $s \in X$ *we have* $t_1^{\mathcal{M}}\langle s\rangle \neq t_2^{\mathcal{M}}\langle s\rangle$.

TL3 $\mathcal{M} \models_X =(t_1, ..., t_n)$ *iff for all* $s, s' \in X$ *such that* $t_1^{\mathcal{M}}\langle s\rangle = t_1^{\mathcal{M}}\langle s'\rangle, ..., t_{n-1}^{\mathcal{M}}\langle s\rangle = t_{n-1}^{\mathcal{M}}\langle s'\rangle$, *we have* $t_n^{\mathcal{M}}\langle s\rangle = t_n^{\mathcal{M}}\langle s'\rangle$.

TL4 $\mathcal{M} \models_X \neg=(t_1, ..., t_n)$ *iff* $X = \emptyset$.

TL5 $\mathcal{M} \models_X Rt_1...t_n$ *iff for all* $s \in X$ *we have* $(t_1^{\mathcal{M}}\langle s\rangle, ..., t_n^{\mathcal{M}}\langle s\rangle) \in R^{\mathcal{M}}$.

TL6 $\mathcal{M} \models_X \neg Rt_1...t_n$ *iff for all* $s \in X$ *we have* $(t_1^{\mathcal{M}}\langle s\rangle, ..., t_n^{\mathcal{M}}\langle s\rangle) \notin R^{\mathcal{M}}$.

TL7 $\mathcal{M} \models_X \phi \otimes \psi$ *iff* $X = Y \cup Z$ *such that* $\mathrm{dom}(Y) = \mathrm{dom}(Z)$, $\mathcal{M} \models_Y \phi$ *and* $\mathcal{M} \models_Z \psi$.

TL8 $\mathcal{M} \models_X \phi \wedge \psi$ *iff* $\mathcal{M} \models_X \phi$ *and* $\mathcal{M} \models_X \psi$.

TL9 $\mathcal{M} \models_X \exists x_n \phi$ *iff* $\mathcal{M} \models_{X(F/x_n)} \phi$ *for some* $F : X \to M$.

TL10 $\mathcal{M} \models_X !x_n \phi$ *iff* $\mathcal{M} \models_{X(M/x_n)} \phi$

TL11 $\mathcal{M} \models_X \sim\phi$ *iff* $\mathcal{M} \not\models_X \phi$.

We can easily define a translation $\phi \mapsto \phi^*$ of dependence logic into team logic, but we have to assume the formula ϕ of dependence logic is in negation normal form:

$$\begin{array}{lcl}
(t = t')^* & = & t = t' \\
(\neg t = t')^* & = & \neg t = t' \\
(Rt_1...t_n)^* & = & Rt_1...t_n \\
(\neg Rt_1...t_n)^* & = & \neg Rt_1...t_n \\
(=(t_1, ..., t_n))^* & = & =(t_1, ..., t_n) \\
(\neg =(t_1, ..., t_n))^* & = & \neg =(t_1, ..., t_n) \\
(\phi \vee \psi)^* & = & \phi^* \otimes \psi^* \\
(\phi \wedge \psi)^* & = & \phi^* \wedge \psi^* \\
(\exists x_n \phi)^* & = & \exists x_n \phi^* \\
(\forall x_n \phi)^* & = & !x_n \phi^*
\end{array}$$

It is an immediate consequence of the definitions that for all $\mathcal{M}$, all formulas ϕ of $\mathcal{D}$, and all X we have $\mathcal{M} \models_X \phi$ in $\mathcal{D}$ if and only if $\mathcal{M} \models_X \phi^*$ in TL. So we may consider $\mathcal{D}$ a fragment of TL, and TL an extension of $\mathcal{D}$ obtained by adding classical negation.

Logical consequence $\phi \Rightarrow \psi$ and logical equivalence $\phi \Leftrightarrow \psi$ are defined similarly as for dependence logic. The following lemma demonstrates that even though $!\, x_n \phi$ for $\phi \in \mathcal{D}$ acts like what is denoted by $\forall x_n \phi$ in dependence logic, it is in the presence of $\sim$ not at all like our familiar universal quantifier, as it commutes with negation:

Lemma 98 $\sim !x_n \phi \Leftrightarrow !x_n \sim \phi$.

Proof. $\mathcal{M} \models_X \sim !x_n\phi$ iff $\mathcal{M} \not\models_X !x_n\phi$ iff $\mathcal{M} \not\models_{X(M/x_n)} \phi$ iff $\mathcal{M} \models_{X(M/x_n)} \sim\phi$, iff $\mathcal{M} \models_X !x_n \sim\phi$. □

We adopt the following abbreviations:

$$
\begin{array}{lcll}
\phi \vee \psi & = & \sim(\sim\phi \wedge \sim\psi) & \text{disjunction} \\
\phi \oplus \psi & = & \sim(\sim\phi \otimes \sim\psi) & \text{sum} \\
\phi \rightarrow \psi & = & \sim\phi \vee \psi & \text{implication} \\
\phi \multimap \psi & = & \sim\phi \oplus \psi & \text{linear implication} \\
\forall x_n\phi & = & \sim\exists x_n \sim\phi & \text{universal quantifier}
\end{array}
$$

Thus $\mathcal{M} \models_X \phi\vee\psi$ iff $\mathcal{M} \models_X \phi$ or $\mathcal{M} \models_X \psi$. Note: $\mathcal{M} \models_X \phi\oplus\psi$ iff whenever
Thus a team of type $\phi \oplus \psi$ has in a sense ϕ or ψ everywhere. We have

$$
\begin{array}{lcl}
\phi\oplus\psi & \Leftrightarrow & \psi\oplus\phi \\
\phi\oplus(\psi\oplus\theta) & \Leftrightarrow & (\phi\oplus\psi)\oplus\theta
\end{array}
$$

but $\phi\oplus\phi \not\Leftrightarrow \phi$. Note also that $\mathcal{M} \models_X \forall x_n\phi$ iff $\mathcal{M} \models_{X(F/x_n)} \phi$ holds for every
Thus a team of type $\forall x_n\phi$ has type ϕ what ever we put as values of x_n.

Remark 99 *The introduction of the new operation $\otimes$ for the operation which in dependence logic used to be denoted $\vee$, and the use of $\vee$ in team logic for the new Boolean disjunction, may seem confusing. The same can be said about the use of $!$ for the operation which in dependence logic used to be denoted $\forall$, while now in team logic $\forall$ has a different meaning. Isn't this quite a mess? However, everything fits into place nicely: The main idea is that ordinary first order formulas, built up from atomic formulas by means of $\sim$, $\vee$, $\wedge$, $\exists$ and $\forall$ have their classical meaning also in team logic: for teams of the form $X = \{s\}$ we have $\mathcal{M} \models_X \phi$ in team logic if and only if $\mathcal{M} \models_s \phi$ in first order logic. This fact is quite robust. We can replace $\sim$ by $\neg$, and $\vee$ by $\otimes$.*

We have

$$
\begin{array}{lcl}
\forall x_n(\phi \wedge \psi) & \Leftrightarrow & (\forall x_n\phi \wedge \forall x_n\psi) \\
\forall x_n\forall x_m\phi & \Leftrightarrow & \forall x_m\forall x_n\phi
\end{array}
$$

$\phi \vee \psi$	the team is either of type ϕ or of type ψ (or both)
$\phi \wedge \psi$	the team is both of type ϕ and of type ψ
$\phi \otimes \psi$	the team divides between type ϕ and type ψ
$\phi \oplus \psi$	every division of the team yields type ϕ or type ψ
$\sim \phi$	the team is not of type ϕ
$\top$	any team
$\perp$	no team
$\mathbf{1}$	any non-empty team
$\mathbf{0}$	only the empty team

Figure 6.1: The intuition behind the logical operations.

but in general (see Lemma 98) $\forall x_n \phi \not\Leftrightarrow\ !\, x_n \phi$.

Definition 100 *The* dependence values *are the following special sentences of team logic:*

$$\begin{array}{rcl} \top & = & =() \\ \perp & = & \sim =() \\ \mathbf{0} & = & \neg =() \\ \mathbf{1} & = & \sim \neg =() \end{array}$$

What about $\neg \sim =()$? This is not a sentence of team logic at all!

Example 101 *Here are some trivial relations between the dependence values:*

1. $\perp \Rightarrow \mathbf{0} \Rightarrow \top$
2. $\perp \Rightarrow \mathbf{1} \Rightarrow \top$
3. $\mathbf{1} = \sim \mathbf{0}, \perp = \sim \top$
4. $\mathbf{0} \Leftrightarrow \sim \mathbf{1}, \top \Leftrightarrow \sim \perp$
5. $\mathbf{0} = \neg \top$

The equation $X = X \cup X$ gives $(\phi \wedge \psi) \Rightarrow (\phi \otimes \psi)$, $(\phi \oplus \psi) \Rightarrow (\phi \vee \psi)$. The equation $X = X \cup \emptyset$ gives $\phi \Rightarrow (\phi \otimes \mathbf{0})$, $(\phi \oplus \perp) \Rightarrow \sim \phi$.

The logic TL is much stronger than $\mathcal{D}$. Let us immediately note the failure of compactness:

Sentence	Teams of that type
$\top$	$\emptyset, \{\emptyset\}$
$\bot$	(none)
$\mathbf{1}$	$\{\emptyset\}$
$\mathbf{0}$	$\emptyset$

Figure 6.2: Dependence values in $\mathcal{D}$.

Proposition 102 *The logic* TL *does not satisfy the Compactness Theorem.*

Proof. Let ϕ_n be the sentence $\forall x_0 ... \forall x_n \exists x_{n+1}(\neg x_0 = x_{n+1} \wedge ... \wedge \neg x_n = x_{n+1})$. Then any finite subset of $T = \{\phi_n : n \in \mathbb{N}\} \cup \{\sim \Phi_\infty\}$ has a model but T itself does not have a model. $\square$

Proposition 103 *The logic* TL *does not satisfy the Löwenheim-Skolem Theorem. There is a sentence ϕ of team logic such that ϕ has an infinite model model, but ϕ has no uncountable models. There is also a sentence ψ of team logic such that ψ has an uncountable model, but no countable models.*

Proof. Recall Lemma 35. Let ϕ be the conjunction of P^- and

$$\begin{array}{rl}\sim \exists x_5 \exists x_4 \,!\, x_0 \exists x_1 \,!\, x_2 \exists x_3 (=\!(x_2, x_3) & \wedge \;\; x_4 < x_5 \\ & \wedge \;\; (((x_0 = x_2 \wedge x_0 < x_5) \wedge \\ & \quad\; (x_1 = x_3 \wedge x_1 < x_4)) \vee \\ & \quad\; ((\neg x_0 = x_2 \vee \neg x_0 < x_5) \wedge \\ & \quad\; (\neg x_1 = x_3 \vee \neg x_1 < x_4)))).\end{array}$$

Then ϕ has the infinite model $(\mathbb{N}, +, \cdot, 0, 1, <)$. But since every model of ϕ is isomorphic to $(\mathbb{N}, +, \cdot, 0, 1, <)$, it cannot have any uncountable models. For ψ recall the sentence Φ_{cmpl} from Section 2.3. Let ψ be the conjunction of the axioms of dense linear order and $\sim \Phi_{\text{cmpl}}$. Now ψ has models, e.g. $(\mathbb{R}, <)$, but every model is a dense complete order and therefore uncountable. $\square$

It follows that there cannot be any translation of team logic into Σ^1_1, as such a translation would yield both the Compactness Theorem and the Löwenheim-Skolem Theorem as a consequence. With a translation to Σ^1_1 ruled out it is difficult to imagine what a game theoretical semantics of team logic would look like.

Note that there cannot be a truth-definition for TL in TL: Suppose $\tau^*(x_0)$ is in TL and $\mathcal{M} \models \phi$ is equivalent to $\mathcal{M} \models \tau^*(\ulcorner\phi\urcorner)$ in all Peano models $\mathcal{M}$. By using the formula $\sim\tau^*(x_0)$ we can reprove Tarski's Undefinability of Truth theorem (Theorem 73).

Despite apparent similarities, team logic and linear logic ([6]) have very little to do with each other. In linear logic resources are split into "consumable" parts. In team logic resources are split into "coherent" parts.

Exercise 118 *Show that every formula of* TL *in which* $\sim$ *does not occur is logically equivalent with a sentence of* $\mathcal{D}$*.*

Exercise 119 *Prove Example 101.*

Exercise 120 *Prove the following equivalences:*

$$\begin{array}{rcl} \phi\wedge\top & \Leftrightarrow & \phi \\ \phi\vee\top & \Leftrightarrow & \top \\ \top\otimes\top & \Leftrightarrow & \top \\ \phi\oplus\top & \Leftrightarrow & \top \end{array}$$

Exercise 121 *Prove the following equivalences:*

$$\begin{array}{rcl} \phi\wedge\bot & \Leftrightarrow & \bot \\ \phi\vee\bot & \Leftrightarrow & \phi \\ \phi\otimes\bot & \Leftrightarrow & \bot \\ \bot\oplus\bot & \Leftrightarrow & \bot \end{array}$$

Exercise 122 *Prove the following equivalences:*

$$\begin{array}{rcl} \phi \wedge \mathbf{0} & \Leftrightarrow & \phi \quad \textit{if } \phi \in \mathcal{D} \\ \phi \vee \mathbf{0} & \Leftrightarrow & \phi \quad \textit{if } \phi \in \mathcal{D} \\ \phi \otimes \mathbf{0} & \Leftrightarrow & \phi \\ \mathbf{0} \oplus \mathbf{0} & \Leftrightarrow & \mathbf{0} \end{array}$$

Exercise 123 *Prove the following equivalences:*

$$\begin{array}{rcl} \mathbf{1} \otimes \mathbf{1} & \Leftrightarrow & \mathbf{1} \\ \mathbf{1} \oplus \mathbf{1} & \Leftrightarrow & \mathbf{1} \end{array}$$

Exercise 124 *Give an example which demonstrates* $\forall x_n \phi \not\Leftrightarrow\, !\, x_n \phi$.

Exercise 125 *Suppose* ϕ *is a formula. Give a formula* ψ *with the property that a team* X *is of type* ψ *if and only if every subset of* X *is of type* ϕ.

Exercise 126 *Suppose* ϕ *is a formula. Give a formula* ψ *with the property that a team* X *is of type* ψ *if and only if every subset of* X *has a subset which is of type* ϕ.

Exercise 127 *Give a formula* ϕ *with the property that a team* X *is of type* ϕ *if and only if every subset of* X *has a subset which is of type* ϕ*, but it is not true that a team* X *is of type* ϕ *if and only if every subset of* X *is of type* ϕ.

Exercise 128 *Show that* $\sim$ *is not definable from the other symbols in team logic, that is, show that the sentence* $\sim P$*, where* P *is a 0-ary predicate symbol, is not logically equivalent to any sentence of team logic of vocabulary* $\{P\}$ *without* $\sim$.

Exercise 129 *Show that* $=(x_1, ..., x_n)$ *is definable from the other symbols in team logic and formulas of the form* $=(t)$*. That is, show that there is a formula* $\phi(x_1, ..., x_n)$ *in team logic such that* $=(x_1, ..., x_n)$ *and* $\phi(x_1, ..., x_n)$ *are logically equivalent and* $\phi(x_1, ..., x_n)$ *has no occurrences of atomic formulas of the form* $=(t_1, ..., t_m)$*, where* $m \geq 2$.

Bibliography

[1] Andreas Blass and Yuri Gurevich. Henkin quantifiers and complete problems. *Ann. Pure Appl. Logic*, 32(1):1–16, 1986.

[2] John P. Burgess. A remark on Henkin sentences and their contraries. *Notre Dame J. Formal Logic*, 44(3):185–188 (electronic) (2004), 2003.

[3] Peter Cameron and Wilfrid Hodges. Some combinatorics of imperfect information. *J. Symbolic Logic*, 66(2):673–684, 2001.

[4] Herbert B. Enderton. Finite partially-ordered quantifiers. *Z. Math. Logik Grundlagen Math.*, 16:393–397, 1970.

[5] David Gale and F. M. Stewart. Infinite games with perfect information. In *Contributions to the theory of games, vol. 2*, Annals of Mathematics Studies, no. 28, pages 245–266. Princeton University Press, Princeton, N. J., 1953.

[6] Jean-Yves Girard. Linear logic. *Theoret. Comput. Sci.*, 50(1):101, 1987.

[7] Kurt Gödel. *Collected works. Vol. I.* The Clarendon Press Oxford University Press, New York, 1986. Publications 1929–1936, Edited and with a preface by Solomon Feferman.

[8] J. Hintikka and G. Sandu. Informational independence as a semantical phenomenon. In *Logic, methodology and philosophy of science, VIII (Moscow, 1987)*, volume 126 of *Stud. Logic Found. Math.*, pages 571–589. North-Holland, Amsterdam, 1989.

[9] J. Hintikka and G. Sandu. Game-theoretic semantics. In *Handbook of Logic and Language*, volume 1, pages 361–410. Elsevier, 1997.

[10] Jaakko Hintikka. *The principles of mathematics revisited.* Cambridge University Press, Cambridge, 1996.

[11] Jaakko Hintikka. Hyperclassical logic (a.k.a. IF logic) and its implications for logical theory. *Bull. Symbolic Logic*, 8(3):404–423, 2002.

[12] Wilfrid Hodges. Compositional semantics for a language of imperfect information. *Log. J. IGPL*, 5(4):539–563 (electronic), 1997.

[13] Wilfrid Hodges. Some strange quantifiers. In *Structures in logic and computer science*, volume 1261 of *Lecture Notes in Comput. Sci.*, pages 51–65. Springer, Berlin, 1997.

[14] Wilfrid Hodges and Jouko Väänänen. Dependence of variables as an atomic formula. (to appear).

[15] Theo M. V. Janssen. Independent choices and the interpretation of IF logic. *J. Logic Lang. Inform.*, 11(3):367–387, 2002. Logic and games.

[16] Thomas Jech. *Set theory.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1997.

[17] S. Krajewski. Mutually inconsistent satisfaction classes. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 22:983–987, 1974.

[18] G. Kreisel and J.-L. Krivine. *Elements of mathematical logic. Model theory.* Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1967.

[19] Saul Kripke. An outline of a theory of truth. *Journal of Philosophy*, 72:690–715, 1975.

[20] Azriel Lévy, A hierarchy of formulas in set theory, *Memoirs of the American Mathematical Society*, 57, 1965, 76 pages.

[21] E. G. K. Lopez-Escobar. A non-interpolation theorem. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 17:109–112, 1969.

[22] Yiannis N. Moschovakis. *Elementary induction on abstract structures*. North-Holland Publishing Co., Amsterdam, 1974. Studies in Logic and the Foundations of Mathematics, Vol. 77.

[23] Gabriel Sandu and Tapani Hyttinen. IF logic and the foundations of mathematics. *Synthese*, 126(1-2):37–47, 2001.

[24] Thoralf Skolem. Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen. *Skrifter utgit av Videnskappsselskapet i Kristiania*, 1920.

[25] Thoralf Skolem. *Selected works in logic*. Edited by Jens Erik Fenstad. Universitetsforlaget, Oslo, 1970.

[26] Alfred Tarski. The semantic conception of truth and the foundations of semantics. *Philos. and Phenomenol. Res.*, 4:341–376, 1944.

[27] Jouko Väänänen. On the semantics of informational independence. *Log. J. IGPL*, 10(3):339–352, 2002.

[28] Jouko Väänänen. A remark on nondeterminacy in IF logic. *Acta Philosophica Fennica*, vol. 78:71–77, 2006.

[29] John von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, New Jersey, 1944.

[30] Wilbur John Walkoe, Jr. Finite partially-ordered quantification. *J. Symbolic Logic*, 35:535–555, 1970.

Index