Models and Games

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Introduction

A recurrent theme in this book is the concept of a game. There are essentially three kinds of games in logic. One is the Semantic Game, also called the Evaluation Game, where the *truth* of a given sentence in a given model is at issue. Another is the Model Existence Game, where the *consistency* in the sense of having a model, or equivalently in the sense of impossibility to derive a contradiction, is at issue. Finally there is the Ehrenfeucht–Fraïssé Game, where *separation* of a model from another by finding a property that is true in one given model but false in another is the goal. The three games are closely linked to each other and one can even say they are essentially variants of just one basic game. This basic game arises from our understanding of the quantifiers. The purpose of this book is to make this strategic aspect of logic perfectly transparent and to show that it underlies not only first-order logic but infinitary logic and logic with generalized quantifiers alike.

We call the close link between the three games the *Strategic Balance of Logic* (Figure 1.1). This balance is perfectly commutative, in the sense that winning strategies can be transferred from one game to another. This mere fact is testimony to the close connection between logic and games, or, thinking semantically, between games and models. This connection arises from the nature of quantifiers. Introducing infinite disjunctions and conjunctions does not upset the balance, barring some set-theoretic issues that may surface. In the last chapter of this book we consider generalized quantifiers and show that the Strategic Balance of Logic persists even in the presence of generalized quantifiers.

The purpose of this book is to present the Strategic Balance of Logic in all its glory.

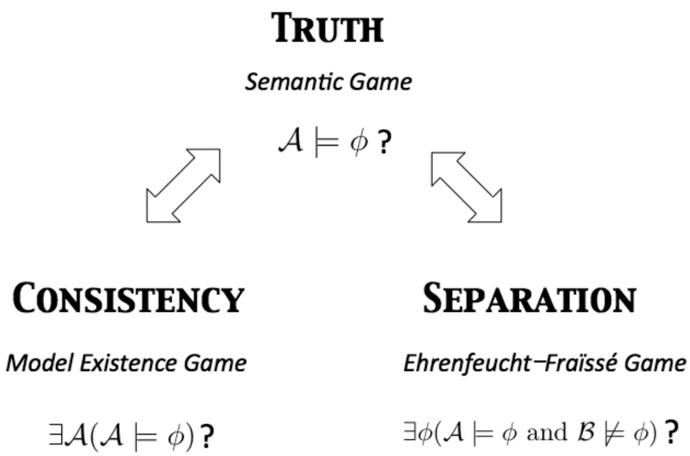

Figure 1.1 The Strategic Balance of Logic.

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Preliminaries and Notation

We use some elementary set theory in this book, mainly basic properties of countable and uncountable sets. We will occasionally use the concept of countable ordinal when we index some uncountable sets. There are many excellent books on elementary set theory. (See Section 2.7.) We give below a simplified account of some basic concepts, the barest outline necessary for this book.

We denote the set $\{0, 1, 2, \ldots\}$ of all natural numbers by $\mathbb{N}$, the set of rational numbers by $\mathbb{Q}$, and the set of all real numbers by $\mathbb{R}$. The power-set operation is written

$$\mathcal{P}(A) = \{B : B \subseteq A\}.$$

We use $A \setminus B$ to denote the set-theoretical difference of the sets A and B. If f is a function, $f''X$ is the set $\{f(x) : x \in X\}$ and $f^{-1}(X)$ is the set $\{x \in \mathrm{dom}(f) : f(x) \in X\}$. *Composition* of two functions f and g is denoted $g \circ f$ and defined by $(g \circ f)(x) = g(f(x))$. We often write fa for $f(a)$. The notation id_A is used for the *identity function* $A \to A$ which maps every element of A to itself, i.e. $id_A(a) = a$ for $a \in A$.

2.1 Finite Sequences

The concept of a finite (ordered) sequence

$$s = (a_0, \ldots, a_{n-1})$$

of elements of a given set A plays an important role in this book. Examples of finite sequences of elements of $\mathbb{N}$ are

$$(8, 3, 9, 67, 200, 0)$$
$$(8, 8, 8)$$

(24).

We can identify the sequence $s = (a_0, \ldots, a_{n-1})$ with the function

$$s' : \{0, \ldots, n-1\} \to A,$$

where

$$s'(i) = a_i.$$

The main property of finite sequences is: $(a_0, \ldots, a_{n-1}) = (b_0, \ldots, b_{m-1})$ if and only if $n = m$ and $a_i = b_i$ for all $i < n$. The number n is called the *length* of the sequence $s = (a_0, \ldots, a_{n-1})$ and is denoted $\text{len}(s)$. A special case is the case $\text{len}(s) = 0$. Then s is called the empty sequence. There is exactly one empty sequence and it is denoted by $\emptyset$.

The Cartesian product of two sets A and B is written

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

More generally

$$A_0 \times \ldots \times A_{n-1} = \{(a_0, \ldots, a_{n-1}) : a_i \in A_i \text{ for all } i < n\}.$$

$$A^n = A \times \ldots \times A \ (n \text{ times}).$$

According to this definition, $A^1 \neq A$. The former consists of sequences of length 1 of elements of A. Note that $A^0 = \{\emptyset\}$.

Finite Sets

A set A is *finite* if it is of the form $\{a_0, \ldots, a_{n-1}\}$ for some natural number n. This means that the set A has at most n elements. If A has exactly n elements we write $|A| = n$ and call $|A|$ the cardinality of A. A set which is not finite is *infinite*. Finite sets form a so-called *ideal*, which means that:

1. $\emptyset$ is finite.
2. If A and B are finite, then so is $A \cup B$.
3. If A is finite and $B \subseteq A$, then also B is finite.

Further useful properties of finite sets are:

1. If A and B are finite, then so is $A \times B$.
2. If A is finite, then so is $\mathcal{P}(A)$.

The *Axiom of Choice* says that for every set A of non-empty sets there is a function f such that $f(a) \in a$ for all $a \in A$. We shall use the Axiom of Choice freely without specifically mentioning it. It needs some practice in set theory to see how the axiom is used. Often an intuitively appealing argument involves a hidden use of it.

Lemma 2.1 *A set A is finite if and only if every injective $f : A \to A$ is a bijection.*

Proof Suppose A is finite and $f : A \to B$ is an injection with $B \subset A$ and $a \in A \setminus B$. Let $a_0 = a$ and $a_{n+1} = f(a_n)$. It is easy to see that $a_n \neq a_m$ whenever $n < m$, so we contradict the finiteness of A. On the other hand, if A is infinite, we can (by using the Axiom of Choice) pick a sequence b_n, $n \in \mathbb{N}$, of distinct elements from A. Then the function g which maps each b_n to b_{n+1} and is the identity elsewhere is an injective mapping from A into A but not a bijection. □

The set of all n-element subsets $\{a_0, \ldots, a_{n-1}\}$ of A is denoted by $[A]^n$.

2.2 Equipollence

Sets A and B are *equipollent*

$$A \sim B$$

if there is a bijection $f : A \to B$. Then $f^{-1} : B \to A$ is a bijection and

$$B \sim A$$

follows. The composition of two bijections is a bijection, whence

$$A \sim B \sim C \Longrightarrow A \sim C.$$

Thus $\sim$ divides sets into equivalence classes. Each equivalence class has a canonical representative (a cardinal number, see the Subsection "Cardinals" below) which is called the *cardinality* of (each of) the sets in the class. The cardinality of A is denoted by $|A|$ and accordingly $A \sim B$ is often written

$$|A| = |B|.$$

One of the basic properties of equipollence is that if

$$A \sim C, B \sim D \text{ and } A \cap B = C \cap D = \emptyset,$$

then

$$A \cup B \sim C \cup D.$$

Indeed, if $f : A \to C$ is a bijection and $g : B \to D$ is a bijection, then $f \cup g : A \cup B \to C \cup D$ is a bijection. If the assumption

$$A \cap B = C \cap D = \emptyset$$

is dropped, the conclusion fails, of course, as we can have $A \cap B = \emptyset$ and $C = D$. It is also interesting to note that even if $A \cap B = C \cap D = \emptyset$, the assumption $A \cup B \sim C \cup D$ does not imply $B \sim D$ even if $A \sim C$ is assumed: Let $A = \mathbb{N}, B = \emptyset, C = \{2n : n \in \mathbb{N}\}$, and $D = \{2n+1 : n \in \mathbb{N}\}$. However, for finite sets this holds: if $A \cup B$ is finite,

$$A \cup B \sim C \cup D,\ A \sim C,\ A \cap B = C \cap D = \emptyset$$

then

$$B \sim D.$$

We can interpret this as follows: the cancellation law holds for finite numbers but does not hold for cardinal numbers of infinite sets.

There are many interesting and non-trivial properties of equipollence that we cannot enter into here. For example the Schröder–Bernstein Theorem: If $A \sim B$ and $B \subseteq C \subseteq A$, then $A \sim C$. Here are some interesting consequences of the Axiom of Choice:

- For all A and B there is C such that $A \sim C \subseteq B$ or $B \sim C \subseteq A$.
- For all infinite A we have $A \sim A \times A$.

It is proved in set theory by means of the Axiom of Choice that $|A| \leq |B|$ holds in the above sense if and only if the cardinality $|A|$ of the set A is at most the cardinality $|B|$ of the set B. Thus the notation $|A| \leq |B|$ is very appropriate.

2.3 Countable sets

A set A which is empty or of the form $\{a_0, a_1, \ldots\}$, i.e. $\{a_n : n \in \mathbb{N}\}$, is called *countable*. A set which is not countable is called *uncountable*. The countable sets form an ideal just as the finite sets do. We now prove two important results about countability. Both are due to Georg Cantor:

Theorem 2.2 *If A and B are countable, then so is $A \times B$.*

Proof If either set is empty, the Cartesian product is empty. So let us assume the sets are both non-empty. Suppose $A = \{a_0, a_1, \ldots\}$ and $B = \{b_0, b_1, \ldots\}$. Let

$$c_n = \begin{cases} (a_i, b_j), & \text{if } n = 2^i 3^j \\ (a_0, b_0), & \text{otherwise.} \end{cases}$$

Now $A \times B = \{c_n : n \in \mathbb{N}\}$, whence $A \times B$ is countable. □

Theorem 2.3 *The union of a countable family of countable sets is countable.*

Proof The empty sets do not contribute anything to the union, so let us assume all the sets are non-empty. Suppose A_n is countable for each $n \in \mathbb{N}$, say, $A_n = \{a^n_m : m \in \mathbb{N}\}$ (we use here the Axiom of Choice to choose an enumeration for each A_n). Let $B = \bigcup_n A_n$. We want to represent B in the form $\{b_n : n \in \mathbb{N}\}$. If n is given, we consider two cases: If n is $2^i 3^j$ for some i and j, we let $b_n = a^i_j$. Otherwise we let $b_n = a^0_0$. □

Theorem 2.4 *The power-set of an infinite set is uncountable.*

Proof Suppose A is infinite and $\mathcal{P}(A) = \{b_n : n \in \mathbb{N}\}$. Since A is infinite, we can choose distinct elements $\{a_n : n \in \mathbb{N}\}$ from A. (This uses the Axiom of Choice. For an argument which avoids the Axiom of Choice see Exercise 2.14.) Let

$$B = \{a_n : a_n \notin b_n\}.$$

Since $B \subseteq A$, there is some n such that $B = b_n$. Is a_n an element of B or not? If it is, then $a_n \notin b_n$ which is a contradiction. So it is not. But then $a_n \in b_n = B$, again a contradiction. □

2.4 Ordinals

The ordinal numbers introduced by Cantor are a marvelous general theory of measuring the *potentially infinite*. They are intimately related to inductive definitions and occur therefore widely in logic. It is easiest to understand ordinals in the context of games, although this was not Cantor's way. Suppose we have a game with two players **I** and **II**. It does not matter what the game is, but it could be something like chess. If **II** can force a win in n moves we say that the game has *rank* n. Suppose then **II** cannot force a win in n moves for any n, but after she has seen the first move of **I**, she can fix a number n and say that she can force a win in n moves. This situation is clearly different from being able to say in advance what n is. So we invent a symbol ω for the rank of this game. In a clear sense ω is greater than each n but there does not seem to be any possible rank between all the finite numbers n and ω. We can think of ω as an infinite number. However, there is nothing metaphysical about the infiniteness of ω. It just has infinitely many predecessors. We can think of ω as a tree T_ω with a root and a separate branch of length n for each n above the root as in the tree on the left in Figure 2.1.

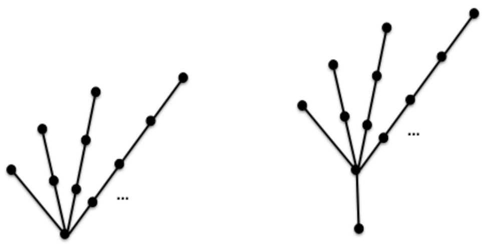


Figure 2.1 T_ω and $T_{\omega+1}$.

Suppose then **II** is not able to declare after the first move how many moves she needs to beat **II**, but she knows how to play her first move in such a way that after **I** has played his second move, she can declare that she can win in n moves. We say that the game has rank $\omega + 1$ and agree that this is greater than ω but there is no rank between them. We can think of $\omega + 1$ as the tree which has a root and then above the root the tree T_ω, as in the tree on the right in Figure 2.1. We can go on like this and define the ranks $\omega + n$ for all n.

Suppose now the rank of the game is not any of the above ranks $\omega + n$, but still **II** can make an interesting declaration: she says that after the first move of **I** she can declare a number m so that after m moves she declares another number n and then in n moves she can force a win. We would say that the rank of the game is $\omega + \omega$. We can continue in this way defining ranks of games that are always finite but potentially infinite. These ranks are what set theorists call ordinals.

We do not give an exact definition of the concept of an ordinal, because it would take us too far afield and there are excellent textbooks on the topic. Let us just note that the key properties of ordinals and their total order $<$ are:

1. Natural numbers are ordinals.
2. For every ordinal α there is an immediate successor $\alpha + 1$.
3. Every non-empty set of ordinals has a smallest element.
4. Every non-empty set of ordinals has a supremum (i.e. a smallest upper bound).

The supremum of the set $\{0, 1, 2, 3, \ldots\}$ of ordinals is denoted by ω. An ordinal is said to be *countable* if it has only countably many predecessors, otherwise *uncountable*. The supremum of all countable ordinals is denoted by ω_1. Here is a picture of the ordinal number "line":

$$0 < 1 < 2 < \ldots < \omega < \omega + 1 < \ldots < \alpha < \alpha + 1 < \ldots < \omega_1 < \ldots$$

Ordinals that have a last element, i.e. are of the form $\alpha + 1$, are called *successor* ordinals; the rest are *limit* ordinals, like ω and $\omega + \omega$.

Ordinals are often used to index elements of uncountable sets. For example, $\{a_\alpha : \alpha < \beta\}$ denotes a set whose elements have been indexed by the ordinal β, called the *length* of the sequence. The set of all such sequences of length β of elements of a given set A is denoted by A^β. The set of all sequences of length $< \beta$ of elements of a given set A is denoted by $A^{<\beta}$.

2.5 Cardinals

Historically cardinals (or more exactly cardinal numbers) are just representatives of equivalence classes of equipollence. Thus there is a cardinal number for countable sets, denoted $\aleph_0$, a cardinal number for the set of all reals, denoted $\mathfrak{c}$, and so on. There is some question as to what exactly are these cardinal numbers. The Axiom of Choice offers an easy answer, which is the prevailing one, as it says that every set can be well-ordered. Then we can let the cardinal number of a set be the order-type of the smallest well-order equipollent with the set. Equivalently, the cardinal number of a set is the smallest ordinal equipollent with the set. If we leave aside the Axiom of Choice, some sets need not have have a cardinal number. However, as is customary in current set theory, let us indeed assume the Axiom of Choice. Then every set has a cardinal number and the cardinal numbers are ordinals, hence well-ordered. The α^{th} infinite cardinal number is denoted $\aleph_\alpha$. Thus $\aleph_1$ is the next in order of magnitude from $\aleph_0$. The famous *Continuum Hypothesis* is the statement that $\aleph_1 = \mathfrak{c}$.

For every set A there exists (by the Axiom of Choice) an ordinal α such that the elements of A can be listed as $\{a_\beta : \beta < \alpha\}$. The smallest such α is called the *cardinal number*, or *cardinality*, of A and denoted by $|A|$. Thus certain ordinals are cardinal numbers of sets. Such ordinals are called *cardinals*. They are considered as canonical representatives of each equivalence class of equipollent sets. For example, all finite numbers are cardinals, as are ω and ω_1. The smallest cardinal such that the smaller infinite cardinals can be enumerated in increasing order as κ_β, $\beta < \alpha$, is denoted ω_α, or alternatively $\aleph_\alpha$. If

$\kappa = \aleph_\alpha$, then $\aleph_{\alpha+1}$ is denoted κ^+ and is called a *successor cardinal*. Cardinals that are not successor cardinals are called *limit cardinals*.

Arithmetic operations $\kappa + \lambda, \kappa \cdot \lambda, \kappa^\lambda$ for cardinals are defined as follows:

$$\kappa + \lambda = |\kappa \cup \lambda|, \ \kappa \cdot \lambda = |\kappa \times \lambda|.$$

Moreover, exponentiation κ^λ of cardinal numbers is defined as the cardinality of the set κ^λ of sequences of elements of κ of length λ. A certain amount of knowledge about the arithmetic of cardinal numbers in necessary in this book, especially in the later chapters, and Chapters 8 and 9 in particular.

The *cofinality* of an ordinal α is the smallest ordinal β for which there is a function $f : \beta \to \alpha$ such that (1) $\xi < \zeta < \beta$ implies $f(\xi) < f(\zeta)$, and (2) for all $\xi < \alpha$ there is some $\zeta < \beta$ such that $\xi < f(\zeta)$. We use $\mathrm{cf}(\alpha)$ to denote the cofinality of α. A cardinal κ is said to be *regular* if $\mathrm{cf}(\kappa) = \kappa$, and *singular* if $\mathrm{cf}(\kappa) < \kappa$. Successor cardinals are always regular. The smallest singular cardinal is $\aleph_\omega$.

The *Continuum Hypothesis* (CH) is the hypothesis $|\mathcal{P}(\mathbb{N})| = \aleph_1$. Neither it nor its negation can be derived from the usual *Zermelo–Fraenkel axioms* of set theory and therefore it (or its negation), like many other similar hypotheses, has to be explicitly mentioned as an assumption, when it is used.

2.6 Axiom of Choice

We have already mentioned the Axiom of Choice. There are so many equivalent formulations of this axiom that books have been written about it. The most notable formulation is the Well-Ordering Principle: every set is equipollent with an ordinal. The Axiom of Choice is sometimes debated because it brings arbitrariness or abstractness into mathematics, often with examples that can be justifiably called pathological, like the Banach–Tarski Paradox: The unit sphere in three-dimensional space can be split into five pieces so that if the pieces are rigidly moved and rotated they form two spheres, each of the original size. The trick is that the splitting exists only in the abstract world of mathematics and can never actually materialize in the physical world. Conclusion: infinite abstract objects do not obey the rules we are used to among finite concrete objects. This is like the situation with sub-atomic elementary particles, where counter-intuitive phenomena, such as entanglement, occur.

Because of the abstractness brought about by the Axiom of Choice it has received criticism and some authors always mention explicitly if they use it in their work. The main problem in working *without* the Axiom of Choice is

3

Games

3.1 Introduction

In this first part we march through the mathematical details of zero-sum two-person games of perfect information in order to be well prepared for the introduction of the three games of the Strategic Balance of Logic (see Figure 1.1) in the subsequent parts of the book. Games are useful as intuitive guides in proofs and constructions but it is also important to know how to make the intuitive arguments and concepts mathematically exact.

3.2 Two-Person Games of Perfect Information

Two-person games of perfect information are like chess: two players set their wits against each other with no role for chance. One wins and the other loses. Everything is out in the open, and the winner wins simply by having a better strategy than the loser.

A Preliminary Example: Nim

In the game of Nim, if it is simplified to the extreme, there are two players **I** and **II** and a pile of six identical tokens. During each round of the game player **I** first removes one or two tokens from the top of the pile and then player **II** does the same, if any tokens are left. Obviously there can be at most three rounds. The player who removes the last token wins and the other one loses.

The game of Figure 3.1 is an example of a zero-sum two-person game of perfect information. It is zero-sum because the victory of one player is the loss of the other. It is of perfect information because both players know what the other player has played. A moment's reflection reveals that player **II** has a way

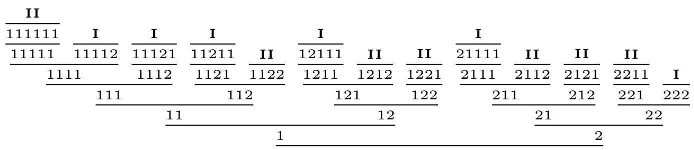


Figure 3.3

cabulary $L = \{W\}$, where W is a four-place predicate symbol. Let $\mathcal{M}$ be an L-structure[2] with $M = \{1, 2\}$ and

$$W^{\mathcal{M}} = \{(a_0, b_0, a_1, b_1) \in M^4 : a_0 + b_0 + a_1 + b_1 = 6\}.$$

Now we have just proved

$$\mathcal{M} \models \forall x_0 \exists y_0 \forall x_1 \exists y_1 W(x_0, y_0, x_1, y_1). \tag{3.1}$$

Conversely, if $\mathcal{M}$ is an arbitrary L-structure, condition (3.1) defines *some* game, maybe not a very interesting one but a game nonetheless: Player **I** picks an element $a_0 \in M$, then player **II** picks an element $b_0 \in M$. Then the same is repeated: player **I** picks an element $a_1 \in M$, then player **II** picks an element $b_1 \in M$. After this player **II** is declared the winner if $(a_0, b_0, a_1, b_1) \in W^{\mathcal{M}}$, and otherwise player **I** is the winner. By varying the structure $\mathcal{M}$ we can model in this way various two-person two-round games of perfect information. This gives a first hint of the connection between games and logic.

Games – a more general formulation

Above we saw an example of a two-person game of perfect information. This concept is fundamental in this book. In general, the simplest formulation of such a game is as follows (see Figure 3.4): There are two players[3] **I** and **II**, a domain A, and a natural number n representing the length of the game. Player **I** starts the game by choosing some element $x_0 \in A$. Then player **II** chooses $y_0 \in A$. After x_i and y_i have been played, and $i + 1 < n$, player **I** chooses $x_{i+1} \in A$ and then player **II** chooses $y_{i+1} \in A$. After n rounds the game ends. To decide who wins we fix beforehand a set $W \subseteq A^{2n}$ of sequences

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \tag{3.2}$$

[2] For the definition of an L-structure see Definition 5.1.

[3] There are various names in the literature for player **I** and **II**, such as player **I** and player **II**, spoiler and duplicator, Nature and myself, or Abelard and Eloise.

I	II
x_0	
	y_0
x_1	
	y_1
$\vdots$	
	$\vdots$
x_{n-1}	
	y_{n-1}

Figure 3.4 A game.

and declare that player **II** wins the game if the sequence formed during the game is in W; otherwise player **I** wins. We denote this game by $\mathcal{G}_n(A, W)$. For example, if $W = \emptyset$, player **II** cannot possibly win, and if $W = A^{2n}$, player **I** cannot possibly win. If W is a set of sequences $(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ where $x_0 = x_1$ and if moreover A has at least two elements, then **II** could not possibly win, as she cannot prevent player **I** from playing x_0 and x_1 differently. On the other hand, W could be the set of all sequences (3.2) such that $y_0 = y_1$. Then **II** can always win because all she has to do during the game is make sure that she chooses y_0 and y_1 to be the same element.

If player **II** has a way of playing that guarantees a sure win, i.e. the opponent **I** loses whatever moves he makes, we say that player **II** has a winning strategy in the game. Likewise, if player **I** has a way of playing that guarantees a sure win, i.e. player **II** loses whatever moves she makes, we say that player **I** has a winning strategy in the game. To make intuitive concepts, such as "way of playing" more exact in the next chapter we define the basic concepts of game theory in a purely mathematical way.

Example 3.1 The game of Nim presented in the previous chapter is in the present notation $\mathcal{G}_3(\{1, 2\}, W)$, where

$$W = \left\{ (a_0, b_0, a_1, b_1, a_2, b_2) \in \{1, 2\}^6 : \sum_{i=0}^{n} (a_i + b_i) = 6 \text{ for some } n \leq 2 \right\}.$$

We allow three rounds as theoretically the players could play three rounds even if player **II** can force a win in two rounds.

Example 3.2 Consider the following game on a set A of integers:

Example 3.5 The following game has no moves:

I	**II**

If $W = \{\emptyset\}$, player **II** is the winner. If $W = \emptyset$, player **I** is the winner. So this is a game with 0 rounds. In practice one of the players would find these games unfair as he or she loses without even having a chance to make a move. It is like being invited to play a game of chess starting in a position where you are already in check-mate.

3.3 The Mathematical Concept of Game

Let A be an arbitrary set and n a natural number. Let $W \subseteq A^{2n}$. We redefine the game

$$\mathcal{G}_n(A, W)$$

in a purely mathematical way. Let us fix two players **I** and **II**. A *play* of one of the players is any sequence $\bar{x} = (x_0, \dots, x_{n-1})$ of elements of A. A sequence

$$(\bar{x}; \bar{y}) = (x_0, y_0, \dots, x_{n-1}, y_{n-1}),$$

of elements of A is called a *play* (of $\mathcal{G}_n(A, W)$). So we have defined the concept of play without any reference to playing the game as an act. The play $(\bar{x}; \bar{y})$ is a *win for* player **II** if

$$(x_0, y_0, \dots, x_{n-1}, y_{n-1}) \in W$$

and otherwise a *win for* player **I**.

Example 3.6 Let us consider the game of chess in this mathematical framework. We modify the game so that the number of rounds is for simplicity exactly n and Black wins a draw, i.e. if neither player has check-mated the other player during those up to n rounds. If a check-mate is reached the rest of the n-round game is of course irrelevant and we can think that the game is finished with “dummy” moves. Let A be the set of all possible positions, i.e. configurations of the pieces on the board. A play $\bar{x}$ of **I** (White) is the sequence of positions where White has just moved. A play $\bar{y}$ of **II** is the sequence of positions where Black has just moved. We let W be the set of plays $(\bar{x}; \bar{y})$, where either White has not obeyed the rules, or Black has obeyed the rules and White has not check-mated Black. With the said modifications, chess is just the game $\mathcal{G}_n(A, W)$ with White playing as player **I** and Black playing as player **II**.

A *strategy* of player **I** in the game $\mathcal{G}_n(A, W)$ is a sequence

$$\sigma = (\sigma_0, \ldots, \sigma_{n-1})$$

of functions $\sigma_i : A^i \to A$. We say that player **I** has *used the strategy* σ *in the play* $(\bar{x}; \bar{y})$ if for all $0 < i < n$:

$$x_i = \sigma_i(y_0, \ldots, y_{i-1})$$

and

$$x_0 = \sigma_0.$$

The strategy σ of player **I** is a *winning strategy*, if every play where **I** has used σ is a win for player **I**. Note that the strategy depends only on the opponent's moves. It is tacitly assumed that when the function σ_{i+1} is used to determine x_{i+1}, the previous functions $\sigma_0, \ldots, \sigma_i$ were used to determine the previous moves $x_0, \ldots, x_n$. Thus a strategy σ is a winning strategy because of the concerted effect of all the functions $\sigma_0, \ldots, \sigma_{n-1}$.

A *strategy* of player **II** in the game $\mathcal{G}_n(A, W)$ is a sequence

$$\tau = (\tau_0, \ldots, \tau_{n-1})$$

of functions $\tau_i : A^{i+1} \to A$. We say that player **II** has *used the strategy* τ *in the play* $(\bar{x}; \bar{y})$ if for all $i < n$:

$$y_i = \tau_i(x_0, \ldots, x_i).$$

The strategy τ of player **II** is a *winning strategy*, if every play where player **II** has used τ is a win for player **II**. A player who has a winning strategy in $\mathcal{G}_n(A, W)$ is said to *win the game* $\mathcal{G}_n(A, W)$.

3.4 Game Positions

A *position* of the game $\mathcal{G}_n(A, W)$ is any initial segment

$$p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$$

of a play $(\bar{x}; \bar{y})$, where $i \leq n$. Positions have a natural ordering: a position p' extends a position p, if p is an initial segment of p'. Of course, this extension-relation is a partial ordering[4] of the set of all positions, that is, if p' extends p and p'' extends p', then p'' extends p, and if p and p' extend each other, then $p = p'$. The empty sequence $\emptyset$ is the smallest element, and the plays $(\bar{x}; \bar{y})$ are

[4] See Example 5.7 for the definition of partial order. Indeed this is a tree-ordering. See Example 5.8 for the definition of tree-ordering.

maximal elements of this partial ordering. A common problem of games is that the set of all positions is huge.

A *strategy of player* **I** *in position* $p = (x_0, y_0, \dots, x_{i-1}, y_{i-1})$ in the game $\mathcal{G}_n(A, W)$ is a sequence

$$\sigma = (\sigma_0, \dots, \sigma_{n-1-i})$$

of functions $\sigma_j : A^j \to A$. We say that player **I** has *used strategy* σ *after position* p *in the play* $(\bar{x}; \bar{y})$, if $(\bar{x}; \bar{y})$ extends p and for all j with $i < j < n$ we have

$$x_j = \sigma_{j-i}(y_i, \dots, y_{j-1})$$

and

$$x_i = \sigma_0.$$

The strategy σ of player **I** in position p is a *winning strategy in position* p, if every play extending p where player **I** has used σ after position p is a win for player **I**.

A *strategy of player* **II** *in position* p in the game $\mathcal{G}_n(A, W)$ is a sequence

$$\tau = (\tau_0, \dots, \tau_{n-1-i})$$

of functions $\tau_j : A^{j+1} \to A$. We say that player **II** has *used strategy* τ *after position* p *in the play* $(\bar{x}; \bar{y})$ if $(\bar{x}; \bar{y})$ extends p and for all j with $i \le j < n$ we have

$$y_j = \tau_{j-i}(x_i, \dots, x_j).$$

The strategy τ of player **II** in position p is a *winning strategy in position* p, if every play extending p where player **II** has used τ after p is a win for player **II**.

The following important lemma shows that if player **II** has a chance in the beginning, i.e. player **I** does not already have a winning strategy, she has a chance all the way.

Lemma 3.7 (Survival Lemma) *Suppose A is a set, n is a natural number, $W \subseteq A^{2n}$ and $p = (x_0, y_0, \dots, x_{i-1}, y_{i-1})$ is a position in the game $\mathcal{G}_n(A, W)$, with $i < n$. Suppose furthermore that player* **I** *does not have a winning strategy in position p. Then for every $x_i \in A$ there is $y_i \in A$ such that player* **I** *does not have a winning strategy in position $p' = (x_0, y_0, \dots, x_i, y_i)$.*

Proof The proof is by contradiction. The intuition is clear: if player **I** had a smart move x_i so that he has a strategy for winning whatever the response y_i of player **II** is, then we could argue that, contrary to the hypothesis, player **I** had a winning strategy already in position p, as he wins whatever **II** moves. Let us now make this idea more exact. Suppose there were an $x_i \in A$ such that for all $y_i \in A$ player **I** has a winning strategy σ^{y_i} in position $p' = (x_0, y_0, \ldots, x_i, y_i)$. We define a strategy $\sigma = (\sigma_0, \ldots, \sigma_{n-1-i})$ of player **I** in position p as follows: $\sigma_0(\emptyset) = x_i$ and

$$\sigma_{j-i}(y_i, \ldots, y_{j-i}) = \sigma^{y_i}(y_{i+1}, \ldots, y_{j-i}).$$

This is a winning strategy of **I** in position p, contrary to our assumption that none exists. □

The following concept is of fundamental importance in game theory and in applications to logic, in particular:

Definition 3.8 A game is called *determined* if one of the players has a winning strategy. Otherwise the game is *non-determined*.

Virtually all games that one comes across in logic are determined. The following theorem is the crucial fact behind this phenomenon:

Theorem 3.9 (Zermelo) *If A is any set, n is a natural number, and $W \subseteq A^{2n}$, then the game $\mathcal{G}_n(A, W)$ is determined.*

Proof Suppose player **I** has no winning strategy. Then player **II** has a winning strategy based on repeated use of Lemma 3.7. Player **II** notes that in the beginning of the game, that is, in position $\emptyset$, player **I** does not have a winning strategy. Then by the Survival Lemma 3.7 she can, whatever player **I** moves, find a move such that afterwards player **I** still does not have a winning strategy. In short, the strategy of player **II** is to prevent player **I** from having a winning strategy. After n rounds the game ends and player **I** still does not have a winning strategy. That means player **I** has lost and player **II** has won. Let us now make this more precise: We define a strategy

$$\tau = (\tau_0, \ldots, \tau_{n-1})$$

of player **II** in the game $\mathcal{G}_n(A, W)$ as follows: Let a be some arbitrary element of A. By Lemma 3.7 we have for each position $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ in the game $\mathcal{G}_n(A, W)$ such that player **I** does not have a winning strategy in position p and each $x_i \in A$ some $y_i \in A$ such that player **I** does not have a winning strategy in position $p' = (x_0, y_0, \ldots, x_i, y_i)$. Let us denote this y_i by

$$y_i = f(p, x_i).$$

If $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ is a position in which player **I** *does* have a winning strategy, we let $f(p, x_i) = a$. We have defined a function f defined on positions p and elements $x_i \in A$. Let $\tau_0(x_0) = f(\emptyset, x_0)$. Assuming $\tau_0, \ldots, \tau_{i-1}$ have been defined already, let

$$\tau_i(x_0, \ldots, x_i) = f(p, x_i),$$

where

$$p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$$

and

$$\begin{aligned} y_0 &= \tau_0(x_0) \\ y_{i-1} &= \tau_{i-1}(x_0, \ldots, x_{i-1}). \end{aligned}$$

It is easy to see that in every play in which player **II** uses this strategy, every position p is such that player **I** *does not* have a winning strategy in position p. It is also easy to see that this is a winning strategy of player **II**. □

3.5 Infinite Games

The concept of a game is by no means limited to games with just finitely many rounds. Imagine a chess board which extends the usual board left and right without end. Then the chess game could go on for infinitely many rounds without the same configuration of pieces coming up twice. A simple infinite game is one in which two players pick natural numbers each choosing a bigger number, if he or she can, than the opponent. There is no end to this game, since there are infinitely many natural numbers. A third kind of infinite game is the following:

Example 3.10 Suppose A is a set of real numbers on the unit interval. We describe a game we denote by $G(A)$. During the game the players decide the decimal expansion of a real number $r = 0.d_0 d_1 \ldots$ on the interval $[0, 1]$. Player **I** decides the even digits d_{2n} and player **II** the odd digits d_{2n+1}. Player **II** wins if $r \in A$. If A is countable, say $A = \{b_n : n \in \mathbb{N}\}$, player **I** has a winning strategy: during round n he chooses the digit d_{2n} so that $r \neq b_n$. If the complement of A is countable, player **II** wins with the same strategy. What if A and its complement are uncountable? This is a well-known and much studied hard question. (See e.g. Jech (1997).)

I	**II**
x_0	
	y_0
x_1	
	y_1
$\vdots$	$\vdots$

Figure 3.5 An infinite game.

If A is any set, we use $A^{\mathbb{N}}$ to denote infinite sequences

$$(x_0, x_1, \ldots)$$

of elements of A. We can think of such sequences as limits of an increasing sequence

$$(x_0), (x_0, x_1), (x_0, x_1, x_2), \ldots$$

of finite sequences.

Let A be an arbitrary set. Let $W \subseteq A^{\mathbb{N}}$. We define the game

$$\mathcal{G}_\omega(A, W)$$

as follows (see Figure 3.5): An infinite sequence

$$(\bar{x}; \bar{y}) = (x_0, y_0, x_1, y_1, \ldots),$$

of elements of A is called a *play* (of $\mathcal{G}_\omega(A, W)$). A *play* of one of the players is likewise any infinite sequence $\bar{x} = (x_0, x_1, \ldots)$ of elements of A. The play $(\bar{x}; \bar{y})$ is a *win for* player **II** if

$$(x_0, y_0, x_1, y_1, \ldots) \in W$$

and otherwise a *win for* player **I** .

A *strategy of* player **I** in the game $\mathcal{G}_\omega(A, W)$ is an infinite sequence

$$\sigma = (\sigma_0, \sigma_1, \ldots)$$

of functions $\sigma_i : A^i \to A$. We say that player **I** has *used the strategy* σ *in the play* $(\bar{x}; \bar{y})$ if for all $i \in \mathbb{N}$:

$$x_i = \sigma_i(y_0, \ldots, y_{i-1})$$

and

$$x_0 = \sigma_0.$$

The strategy σ of player **I** is a *winning strategy*, if every play where **I** has used σ is a win for player **I**.

A *strategy of* player **II** in the game $\mathcal{G}_\omega(A, W)$ is an infinite sequence

$$\tau = (\tau_0, \tau_1, \ldots)$$

of functions $\tau_i : A^{i+1} \to A$. We say that player **II** has *used the strategy* τ *in the play* $(\bar{x}; \bar{y})$ if for all $i < n$:

$$y_i = \tau_i(x_0, \ldots, x_i).$$

The strategy τ of player **II** is a *winning strategy*, if every play where player **II** has used τ is a win for player **II**. A player is said to *win the game* $\mathcal{G}_\omega(A, W)$ if he or she has a winning strategy in it.

A *position* of the infinite game $\mathcal{G}_\omega(A, W)$ is any initial segment

$$p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$$

of a play $(\bar{x}; \bar{y})$. We say that player **I** has *used strategy* $\sigma = (\sigma_0, \sigma_1, \ldots)$ *after position* p *in the play* $(\bar{x}; \bar{y})$, if $(\bar{x}; \bar{y})$ extends p and for all j with $i < j$ we have $x_j = \sigma_{j-i}(y_i, \ldots, y_{j-1})$ and $x_i = \sigma_0$. The strategy σ of player **I** is a *winning strategy in position* p, if every play extending p where player **I** has used σ after position p is a win for player **I**. We say that player **II** has *used strategy* $\tau = (\tau_0, \tau_1, \ldots)$ *after position* p *in the play* $(\bar{x}; \bar{y})$ if for all j with $i \leq j$ we have $y_j = \tau_{j-i}(x_i, \ldots, x_j)$. The strategy τ of player **II** is a *winning strategy in position* p, if every play extending p where player **II** has used τ after p is a win for player **II**.

An important example of a class of infinite games is the class of *open* or *closed* games of length ω. A subset W of $A^{\mathbb{N}}$ is *open,*[5] if

$$(x_0, y_0, x_1, y_1, \ldots) \in W$$

implies the existence of $n \in \mathbb{N}$ such that

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x'_n, y'_n, x'_{n+1}, y'_{n+1}, \ldots) \in W$$

for all $x'_n, y'_n, x'_{n+1}, y'_{n+1}, \ldots \in A$. Respectively, W is *closed* if $A^{\mathbb{N}} \setminus W$ is open. Finally, W is *clopen* if it is both open and closed. We call a game $G_\omega(A, W)$ closed (or open or clopen) if the set W is. We are mainly concerned in this book with closed games. A typical strategy of player **II** in a closed game is to “hang in there”, as she knows that if player **I** ends up winning the play $p = (x_0, y_0, \ldots)$, that is, $p \notin W$, there is some n such that player **I** won the game already in position $(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$.

[5] The collection of open subsets of $A^{\mathbb{N}}$ is a topology, hence the name.

We can think of infinite games as *limits* of finite games as follows: Any finite game $G_n(A, W)$ can be made infinite by disregarding the moves after the usual n moves. The resulting infinite game is clopen (see Exercise 3.31). On the other hand, if $G_\omega(A, W)$ is an infinite game and $n \in \mathbb{N}$ we can form an n-round game by simply considering only the first n rounds of $G_\omega(A, W)$ and declaring a play of n rounds a win for player **II** if *any* infinite play extending it is in W. Unless W is open or closed, there may be very little connection between the resulting finite games and the original infinite game (see however Exercise 3.32).

Lemma 3.11 (Infinite Survival Lemma) *Suppose A is a set, $W \subseteq A^{\mathbb{N}}$, and $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ is a position in the game $\mathcal{G}_\omega(A, W)$, with $i \in \mathbb{N}$. Suppose furthermore that player* **I** *does not have a winning strategy in position p. Then for every $x_i \in A$ there is $y_i \in A$ such that player* **I** *does not have a winning strategy in position $p' = (x_0, y_0, \ldots, x_i, y_i)$.*

Proof The proof is by contradiction. Suppose there were an $x_i \in A$ such that for all $y_i \in A$ player **I** has a winning strategy σ^{y_i} in position $p' = (x_0, y_0, \ldots, x_i, y_i)$. We define a strategy $\sigma = (\sigma_0, \sigma_1, \ldots)$ of player **I** in position p as follows: $\sigma_0(\emptyset) = x_i$ and for $j > i$,

$$\sigma_{j-i}(y_i, \ldots, y_{j-1}) = \sigma^{y_i}(y_{i+1}, \ldots, y_{j-i}).$$

This is a winning strategy of player **I** in position p, contrary to assumption. □

Theorem 3.12 (Gale–Stewart) *If A is any set and $W \subseteq A^{\mathbb{N}}$ is open or closed, then the game $\mathcal{G}_\omega(A, W)$ is determined.*

Proof Suppose first W is closed and player **I** has no winning strategy. We define a strategy

$$\tau = (\tau_0, \tau_1, \ldots)$$

of player **II** in the game $\mathcal{G}_\omega(A, W)$ as follows: Let a be some arbitrary element of A. By Lemma 3.11 we have for each position $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ in the game $\mathcal{G}_\omega(A, W)$ such that player **I** does not have a winning strategy in position p, and each $x_i \in A$, some $y_i \in A$ such that player **I** does not have a winning strategy in position $p' = (x_0, y_0, \ldots, x_i, y_i)$. Let us denote this y_i by

$$y_i = f(p, x_i).$$

If $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ is a position in which player **I** *does* have a winning strategy, we let $f(p, x_i) = a$. We have defined a function f defined on positions p and elements $x_i \in A$. Let $\tau_0(x_0) = f(\emptyset, x_0)$. Assuming $\tau_0, \ldots, \tau_{i-1}$ have been defined already, let $\tau_i(x_0, \ldots, x_i) = f(p, x_i)$, where

$p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ and $y_0 = \tau_0(x_0), y_{i-1} = \tau_{i-1}(x_0, \ldots, x_{i-1})$. It is easy to see that in every play in which player **II** uses this strategy, every position p is such that player **I** does not have a winning strategy in position p. It is also easy to see that this is a winning strategy of player **II**.

The proof is similar if W is open. It follows that $G_\omega(A, W)$ is determined. □

Theorem 3.12 can been vastly generalized, see e.g. (Jech, 1997, Chapter 33). The *Axiom of Determinacy* says that the game $G_\omega(A, W)$ is determined for all sets A and W. However, this axiom contradicts the Axiom of Choice. By using the Axiom of Choice one can show that there are sets A of real numbers such that the game $G(A)$ is not determined (see Exercise 3.37).

3.6 Historical Remarks and References

The mathematical theory of games was started by von Neumann and Morgenstern (1944). For the early history of two-person zero-sum games of perfect information, see Schwalbe and Walker (2001). See Mycielski (1992) for a more recent survey on games of perfect information. Theorem 3.12 goes back to Gale and Stewart (1953).

Exercises

3.1 Consider the following game: Player **I** picks a natural number n. Then player **II** picks a natural number m. If $2^m = n$, then **II** wins, otherwise **I** wins. Express this game in the form $\mathcal{G}_1(A, W)$.

3.2 Consider the following game: Player **I** picks a natural number n. Then player **II** picks two natural numbers m and k. If $m \cdot k = n$, then **II** wins, otherwise **I** wins. Express this game in the form $\mathcal{G}_2(A, W)$.

3.3 Consider $\mathcal{G}_3(A, W)$, where $A = \{0, 1, 2\}$ and

1. $W = \{(x_0, y_0, x_1, y_1, x_2, y_2) \in A^3 : x_0 = y_2\}$.
2. $W = \{(x_0, y_0, x_1, y_1, x_2, y_2) \in A^3 : y_0 \neq x_2$ or $y_2 \neq x_0\}$.
3. $W = \{(x_0, y_0, x_1, y_1, x_2, y_2) \in A^3 : x_0 \neq y_2$ and $x_1 \neq y_2$ and $x_2 \neq y_2\}$.

Who has a winning strategy?

3.4 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a mapping. Express the condition that f is uniformly continuous as a game and as the truth of a first-order sentence in a suitable structure.

3.10 Examine the game determined by condition (3.1) $M = \mathbb{N}$ and $W^{\mathcal{M}} = \{(a_0, b_0, a_1, b_1) \in M^4 : a_0 < b_0$ and either a_1 does not divide b_0 or $b_1 = a_1 = 1$ or $b_1 = a_1 = b_0\}$. Who has a winning strategy?

3.11 Suppose X is a set of positions of the game $G_n(A, W)$ such that

1. $\emptyset \in X$.
2. For all $i < n$, all $(x_0, y_0, \dots, x_{i-1}, y_{i-1}) \in X$, and all $x_i \in A$ there is $y_i \in A$ such that $(x_0, y_0, \dots, x_i, y_i) \in X$.
3. If $p = (x_0, y_0, \dots, x_{n-1}, y_{n-1}) \in X$, then $p \in W$.

Show that player **II** has a winning strategy in the game $G_n(A, W)$. Give such a set for the game of Example 3.1.

3.12 Suppose that player **II** has a winning strategy in the game $G_n(A, W)$. Show that there is a set X of positions of the game $G_n(A, W)$ satisfying conditions 1–3 of the previous exercise.

3.13 Suppose X is a set of positions of the game $G_n(A, W)$ such that

1. $\emptyset \in X$.
2. For all $i < n$, all $(x_0, y_0, \dots, x_{i-1}, y_{i-1}) \in X$ there is $x_i \in A$ such that for all $y_i \in A$ we have $(x_0, y_0, \dots, x_i, y_i) \in X$.
3. If $p = (x_0, y_0, \dots, x_{n-1}, y_{n-1}) \in X$, then $p \notin W$.

Show that player **I** has a winning strategy in the game $G_n(A, W)$. Give such a set for the game of Example 3.1 when we start with seven tokens.

3.14 Suppose that player **I** has a winning strategy in the game $G_n(A, W)$. Show that there is a set X of positions of the game $G_n(A, W)$ satisfying conditions 1–3 of the previous exercise.

3.15 Suppose A is finite. Describe an algorithm which searches for a winning strategy for a player in $\mathcal{G}_n(A, W)$, provided the player has one.

3.16 Finish the proof of Lemma 3.7 by showing that the strategy described in the proof is indeed a winning strategy of player **I**.

3.17 Finish the proof of Theorem 3.9 by showing that the strategy described in the proof is indeed a winning strategy of player **II**.

3.18 Consider $\mathcal{G}_2(A, W)$, where $A = \{0, 1\}$ and

1. $W = \{(x_0, y_0, x_1, y_1) \in A^2 : x_0 = y_1\}$.
2. $W = \{(x_0, y_0, x_1, y_1) \in A^2 : y_0 \neq x_1$ or $y_1 \neq x_0\}$.
3. $W = \{(x_0, y_0, x_1, y_1) \in A^2 : x_0 \neq y_1$ and $x_1 \neq y_1\}$.

In each case give a winning strategy for one of the players.

3.19 Suppose σ is a strategy of player **I** and τ a strategy of player **II** in $\mathcal{G}_n(A, W)$. Show that there is exactly one play $(\bar{x}; \bar{y})$ of $\mathcal{G}_n(A, W)$ such that player **I** has used σ and player **II** has used τ in it.

3.20 Show that at most one player can have a winning strategy in $\mathcal{G}_n(A, W)$.

3.21 Give the winning strategy of player **II** in Nim (Example 3.1) in the form $\tau = (\tau_0, \tau_1)$.

3.22 Consider the game of Example 3.3 when $f(x) = 2x + 3$, $a \in \mathbb{R}$, and $b = 2a + 3$. Give some winning strategy of player **II**.

3.23 Consider the game of Example 3.3 when $f(x) = 2x + 3$, $a = 1$ and $b = 4$. Give some winning strategy of player **I**.

3.24 Consider the game of Example 3.3 when $f(x) = x^2$, $a \in \mathbb{R}$, and $b = a^2$. Give some winning strategy of player **II**.

3.25 A more general version of Nim has m tokens rather than six. Decide who has a winning strategy for each m and give the winning strategy.

3.26 Suppose we have two games $\mathcal{G}_n(A, W)$ and $\mathcal{G}_n(A', W')$, where $A \cap A' = \emptyset$. Let $A'' = A \cup A'$ and let W'' be the set of sequences

$$(x_0, y_0, \ldots, x_{2n-1}, y_{2n-1}),$$

which satisfy the following condition:

$$(x_0, y_0, x_2, y_2 \ldots, x_{2n-2}, y_{n-2}) \in W$$

and

$$(x_1, y_1, x_3, y_3, \ldots, x_{2n-1}, y_{2n-1}) \in W'.$$

Show that:

1. If player **I** has a winning strategy in $\mathcal{G}_n(A, W)$ *or* in $\mathcal{G}_n(A', W')$, then he has one in $\mathcal{G}_{2n}(A'', W'')$.
2. If player **II** has a winning strategy in $\mathcal{G}_n(A, W)$ *and* in $\mathcal{G}_n(A', W')$, then she has one in $\mathcal{G}_{2n}(A'', W'')$.

3.27 Suppose we have two games $\mathcal{G}_n(A, W)$ and $\mathcal{G}_n(A', W')$. Let $A'' = A \times A'$ and let W'' be the set of sequences

$$(((x_0, x'_0), (y_0, y'_0)), \ldots, ((x_{n-1}, x'_{n-1}), (y_{n-1}, y'_{n-1}))),$$

where

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in W$$

and

$$(x'_0, y'_0, \ldots, x'_{n-1}, y'_{n-1}) \in W'.$$

Show that:

1. If player **I** has a winning strategy in $\mathcal{G}_n(A, W)$ *or* in $\mathcal{G}_n(A', W')$, then he has one in $\mathcal{G}_n(A'', W'')$.
2. If player **II** has a winning strategy in $\mathcal{G}_n(A, W)$ *and* in $\mathcal{G}_n(A', W')$, then she has one in $\mathcal{G}_n(A'', W'')$.

5
Models

5.1 Introduction

The concept of a model (or structure) is one of the most fundamental in logic. In brief, while the meaning of logical symbols $\wedge, \vee, \exists, \ldots$ is always fixed, models give meaning to non-logical symbols such as constant, predicate, and function symbols. When we have agreed about the meaning of the logical and non-logical symbols of logic, we can then define the meaning of arbitrary formulas.

Depending on context and preference, models appear in logic in two roles. They can serve the auxiliary role of clarifying logical derivation. For example, one quick way to tell what it means for φ to be a logical consequence of ψ is to say that in every model where ψ is true also φ is true. It is then an almost trivial matter to understand why for example $\forall x \exists y \varphi$ is a logical consequence of $\exists y \forall x \varphi$ but $\forall y \exists x \varphi$ is in general not.

Alternatively models can be the prime objects of investigation and it is the logical derivation that is in an auxiliary role of throwing light on properties of models. This is manifestly demonstrated by the Completeness Theorem which says that any set T of first-order sentences has a model unless a contradiction can be logically derived from T, which entails that the two alternative perspectives of models are really equivalent. Since derivations are finite, this implies the important Compactness Theorem: If a set of first-order sentences is such that each of its finite subsets has a model it itself has a model. The Compactness Theorem has led to an abundance of non-isomorphic models of first-order theories, and constitutes the origin of the whole subject of Model Theory. In this chapter models are indeed the prime objects of investigation and we introduce auxiliary concepts such as the Ehrenfeucht–Fraïssé Game that help us understand models.

We use the words “model” and “structure” as synonyms. We have a slight preference for the word "structure" in a context where absolute generality prevails and the structures are not assumed to satisfy any particular axioms. Respectively, our preference is to call a structure that satisfies some given axioms a model, so a structure satisfying a theory is called a model of the theory.

5.2 Basic Concepts

A *vocabulary* is any set L of predicate symbols $P, Q, R, \ldots$, function symbols $f, g, h, \ldots$, and constant symbols $c, d, e, \ldots$. Each vocabulary has an *arity-function*

$$\#_L : L \to \mathbb{N}$$

which tells the arity of each symbol. Thus if $P \in L$, then P is a $\#_L(P)$-ary predicate symbol. If $f \in L$, then f is a $\#_L(f)$-ary function symbol. Finally, $\#_L(c)$ is assumed to be 0 for constants $c \in L$. Predicate or function symbols of arity 1 are called *unary* or *monadic*, and those of arity 2 are called *binary*. A vocabulary is called unary (or binary) if it contains only unary (respectively, binary) symbols. A vocabulary is called *relational* if it contains no function or constant symbols.

Definition 5.1 An L-*structure* (or L-*model*) is a pair $\mathcal{M} = (M, \mathrm{Val}_{\mathcal{M}})$, where M is a non-empty set called the *universe (or the domain)* of $\mathcal{M}$, and $\mathrm{Val}_{\mathcal{M}}$ is a function defined on L with the following properties:

1. If $R \in L$ is a relation symbol and $\#_L(R) = n$, then $\mathrm{Val}_{\mathcal{M}}(R) \subseteq M^n$.
2. If $f \in L$ is a function symbol and $\#_L(f) = n$, then $\mathrm{Val}_{\mathcal{M}}(f) : M^n \to M$.
3. If $c \in L$ is a constant symbol, then $\mathrm{Val}_{\mathcal{M}}(c) \in M$.

We use $\mathrm{Str}(L)$ to denote the class of all L-structures.

We usually shorten $\mathrm{Val}_{\mathcal{M}}(R)$ to $R^{\mathcal{M}}$, $\mathrm{Val}_{\mathcal{M}}(f)$ to $f^{\mathcal{M}}$, and $\mathrm{Val}_{\mathcal{M}}(c)$ to $c^{\mathcal{M}}$. If no confusion arises, we use the notation

$$\mathcal{M} = (M, R_1^{\mathcal{M}}, \ldots, R_n^{\mathcal{M}}, f_1^{\mathcal{M}}, \ldots, f_m^{\mathcal{M}}, c_1^{\mathcal{M}}, \ldots, c_k^{\mathcal{M}})$$

for an L-structure $\mathcal{M}$, where $L = \{R_1, \ldots, R_n, f_1, \ldots, f_m, c_1 \ldots, c_k\}$.

Example 5.2 Graphs are L-structures for the relational vocabulary $L = \{E\}$, where E is a predicate symbol with $\#_L(E) = 2$. Groups are L-structures for $L = \{\circ\}$, where $\circ$ is a binary function symbol. Fields are L-structures for $L = \{+, \cdot, 0, 1\}$, where $+, \cdot$ are binary function symbols and $0, 1$ are constant symbols. Ordered sets (i.e. linear orders) are L-structures for the relational vocabulary $L = \{<\}$, where $<$ is a binary predicate symbol. If $L = \emptyset$, an L-structure (M) is a structure with just the universe and no structure in it.

If $\mathcal{M}$ is a structure and π maps M bijectively onto another set M', we can use π to copy the relations, functions, and constants of $\mathcal{M}$ on M'. In this way we get a perfect copy $\mathcal{M}'$ of $\mathcal{M}$ which differs from $\mathcal{M}$ only in the respect that the underlying elements are different. We then say that $\mathcal{M}'$ is an isomorphic copy of $\mathcal{M}$. For all practical purposes we consider the structures $\mathcal{M}$ and $\mathcal{M}'$ as one and the same structure. However, they are not the same structure, just isomorphic. This may sound as if isomorphism was a rather trivial matter, but this is not true. In many cases it is a highly non-trivial enterprise to investigate whether two structures are isomorphic or not. In the realm of finite structures the question of deciding whether two given structures are isomorphic or not is a famous case of a complexity question which is between P (polynomial time) and NP (non-deterministic polynomial time) and about which we do not know whether it is NP-complete. In the light of present knowledge it is conceivable that this question is strictly between P and NP.

Definition 5.3 L-structures $\mathcal{M}$ and $\mathcal{M}'$ are *isomorphic* if there is a bijection

$$\pi : M \to M'$$

such that

1. For all $a_1, \ldots, a_{\#_L(R)} \in M$:

$$(a_1, \ldots, a_{\#_L(R)}) \in R^{\mathcal{M}} \iff (\pi(a_1), \ldots, \pi(a_{\#_L(R)})) \in R^{\mathcal{M}'}.$$

2. For all $a_1, \ldots, a_{\#_L(f)} \in M$:

$$f^{\mathcal{M}'}(\pi(a_1), \ldots, \pi(a_{\#_L(f)})) = \pi(f^{\mathcal{M}}(a_1, \ldots, a_{\#_L(f)})).$$

3. For all $c \in L$: $\pi(c^{\mathcal{M}}) = c^{\mathcal{M}'}$.

In this case we say that π is an *isomorphism* $\mathcal{M} \to \mathcal{M}'$, denoted

$$\pi : \mathcal{M} \cong \mathcal{M}'.$$

If also $\mathcal{M} = \mathcal{M}'$, we say that π is an *automorphism* of $\mathcal{M}$.

Example 5.4 *Unary* (or *monadic*) structures, i.e. L-structures for unary L, are particularly simple and easy to deal with. Figure 5.1 depicts a unary structure. Suppose L consists of unary predicate symbols $R_1, \ldots, R_n$ and $\mathcal{A}$ is an L-structure. If $X \subseteq A$ and $d \in \{0, 1\}$, let $X^d = X$ if $d = 0$ and $X^d = A \setminus X$

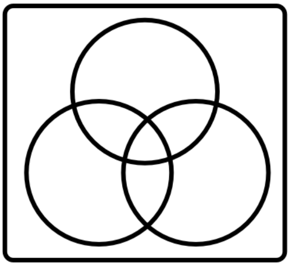


Figure 5.1 A unary structure.

otherwise. Suppose $\epsilon : \{1, \ldots, n\} \to \{0, 1\}$. The *$\epsilon$-constituent* of $\mathcal{A}$ is the set

$$C_\epsilon(\mathcal{A}) = \bigcap_{i=1}^{n} (R_i^{\mathcal{A}})^{\epsilon(i)}.$$

A priori, the 2^n sets $C_\epsilon(\mathcal{A})$ can each have any cardinality whatsoever. It is the nature of unary structures that the constituents are totally independent of each other. If $\mathcal{A} \cong \mathcal{B}$, then

$$|C_\epsilon(\mathcal{A})| = |C_\epsilon(\mathcal{B})| \tag{5.1}$$

for every ϵ. Conversely, if two L-structures $\mathcal{A}$ and $\mathcal{B}$ satisfy Equation (5.1) for every ϵ, then $\mathcal{A} \cong \mathcal{B}$ (see Exercise 5.6). We can say that the function $\epsilon \mapsto |C_\epsilon(\mathcal{A})|$ characterizes completely (i.e. up to isomorphism) the unary structure $\mathcal{A}$. There is nothing more we can say about $\mathcal{A}$ but this function.

Example 5.5 *Equivalence relations*, i.e. L-structures $\mathcal{M}$ for $L = \{\sim\}$ such that $\sim^{\mathcal{M}}$ is a symmetric ($x \sim y \Rightarrow y \sim x$), transitive ($x \sim y \sim z \Rightarrow x \sim z$), and reflexive ($x \sim x$) relation on M can be characterized almost as easily as unary structures. Let for every cardinal number $\kappa \leq |M|$ the number of equivalence classes of $\sim^{\mathcal{M}}$ of cardinality κ be denoted by $EC_\kappa(\mathcal{M})$. If $\mathcal{A} \cong \mathcal{B}$, then

$$EC_\kappa(\mathcal{A}) = EC_\kappa(\mathcal{B}) \tag{5.2}$$

for every $\kappa \leq |A|$. Conversely, if two L-structures $\mathcal{A}$ and $\mathcal{B}$ satisfy Equation (5.2) for every $\kappa \leq |A \cup B|$, then $\mathcal{A} \cong \mathcal{B}$ (see Exercise 5.12). We can say that the function $\kappa \mapsto EC_\kappa(\mathcal{A})$ characterizes completely (i.e. up to isomorphism) the equivalence relation $\mathcal{A}$. There is nothing more we can say about $\mathcal{A}$ but this function. For equivalence relations on a finite universe of size n this

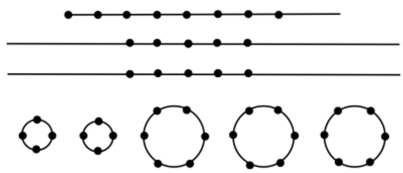


Figure 5.4 A successor structure.

If $\mathcal{M}$ is a successor structure, let $Cmp_{\mathcal{M}}$ be the set of components of $\mathcal{M}$ and

$$CC_n(\mathcal{M}) = |\{C \in Cmp_{\mathcal{M}} : C \text{ is an } n\text{-cycle component}\}|,$$

$$CC_\infty(\mathcal{M}) = |\{C \in Cmp_{\mathcal{M}} : C \text{ is a } \mathbb{Z}\text{-component}\}|.$$

Two successor structures $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if $CC_a(\mathcal{M}) = CC_a(\mathcal{N})$ for all $a \in \mathbb{N} \cup \{\infty\}$.

5.3 Substructures

The concept of a substructure is in principle a very simple one, especially for relational vocabularies. There are however subtleties which deserve special attention when function symbols are involved.

Definition 5.10 An L-structure $\mathcal{M}$ is a *substructure* of another L-structure $\mathcal{M}'$, in symbols $\mathcal{M} \subseteq \mathcal{M}'$, if:

1. $M \subseteq M'$.
2. $R^{\mathcal{M}} = R^{\mathcal{M}'} \cap M^n$ if $R \in L$ is an n-ary predicate symbol.
3. $f^{\mathcal{M}} = f^{\mathcal{M}'} \restriction M^n$ if $f \in L$ is an n-ary function symbol.
4. $c^{\mathcal{M}} = c^{\mathcal{M}'}$ if $c \in L$ is a constant symbol.

Substructures are particularly easy to understand in the case that L is relational. Then any subset M of an L-structure $\mathcal{M}'$ determines a substructure $\mathcal{M}$ the universe of which is M. If L is not relational we have to worry about the question whether M is closed under the functions $f^{\mathcal{M}'}$, $f \in L$, and whether the interpretations $c^{\mathcal{M}'}$ of constant symbols $c \in L$ are in M. For example, if $L = \{f\}$ where f is a unary function symbol, then any substructure of an L-structure which contains an element a has to contain also $f^{\mathcal{M}'}(a), f^{\mathcal{M}'}(f^{\mathcal{M}'}(a))$, etc. A substructure of a group need not be a subgroup even when it is closed under the group operation. For example, $(\mathbb{N}, +)$ is a substructure of $(\mathbb{Z}, +)$ but it is not a group. A substructure of a linear order is again a linear order. Similarly, a substructure of a partial order is again a partial order. A substructure of a tree is a tree if it has a smallest element.

Lemma 5.11 *Suppose L is a vocabulary, $\mathcal{M}$ an L-structure, and $X \subseteq M$. Suppose furthermore that either L contains constant symbols or $X \neq \emptyset$. There is a unique L-structure $\mathcal{N}$ such that:*

1. *$\mathcal{N} \subseteq \mathcal{M}$.*
2. *$X \subseteq N$.*
3. *If $\mathcal{N}' \subseteq \mathcal{M}$ and $X \subseteq N'$, then $\mathcal{N} \subseteq \mathcal{N}'$.*

Proof Let $X_0 = X \cup \{c^{\mathcal{M}} : c \in L\}$ and inductively

$$X_{n+1} = X_n \cup \{f^{\mathcal{M}}(a_1, \dots, a_{\#_L(f)}) : a_1, \dots, a_{\#_L(f)} \in X_n, f \in L\}.$$

It is easy to see that the set $N = \bigcup_{n \in \mathbb{N}} X_n$ is the universe of the unique structure $\mathcal{N}$ claimed to exist in the lemma. □

We call the unique structure $\mathcal{N}$ of Lemma 5.11 the substructure of $\mathcal{M}$ *generated* by X and denote it by $[X]_{\mathcal{M}}$. The following lemma is used repeatedly in the sequel.

Lemma 5.12 *Suppose L is a vocabulary. Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures and $\pi : M \to N$ is a partial mapping. There is at most one isomorphism $\pi^* : [\mathrm{dom}(\pi)]_{\mathcal{M}} \to [\mathrm{rng}(\pi)]_{\mathcal{N}}$ extending π.*

5.4 Back-and-Forth Sets

One of the main themes of this book is the question: Given two structures $\mathcal{M}$ and $\mathcal{N}$, how do we measure how close they are to being isomorphic? They may be non-isomorphic for a totally obvious reason, e.g. two graphs one of which has a triangle while the other does not. They may also be non-isomorphic for an extremely subtle reason which involves the use of the Axiom of Choice (see e.g. Lemma 9.9). One of the basic tools in trying to answer this question is the concept of partial isomorphism.

Definition 5.13 Suppose L is a vocabulary and $\mathcal{M}, \mathcal{M}'$ are L-structures. A partial mapping $\pi : M \to M'$ is a *partial isomorphism* $\mathcal{M} \to \mathcal{M}'$ if there is an isomorphism $\pi^* : [\mathrm{dom}(\pi)]_{\mathcal{M}} \to [\mathrm{rng}(\pi)]_{\mathcal{M}'}$ extending π. We use $\mathrm{Part}(\mathcal{M}, \mathcal{M}')$ to denote the set of partial isomorphisms $\mathcal{M} \to \mathcal{M}'$. If $\mathcal{M} = \mathcal{M}'$ we call π a *partial automorphism*.

Note that the extension π^* referred to in Definition 5.13 is by Lemma 5.12 necessarily unique.

The main topic of this section, the back-and-forth sets, are very useful weaker versions of isomorphisms. To get a picture of this, suppose $f : \mathcal{A} \cong \mathcal{B}$. Then $f \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$ and we can go back and forth between $\mathcal{A}$ and $\mathcal{B}$ with f in the following sense:

$$\forall a \in A \exists b \in B (f(a) = b) \tag{5.6}$$

$$\forall b \in B \exists a \in A (f(a) = b). \tag{5.7}$$

We now generalize this to a situation where we do not quite have an isomorphism but only a set P which reflects the back and forth conditions (5.8) and (5.9) of an isomorphism.

Definition 5.14 Suppose $\mathcal{A}$ and $\mathcal{B}$ are L-structures. A *back-and-forth set* for $\mathcal{A}$ and $\mathcal{B}$ is any non-empty set $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B})$ such that

$$\forall f \in P \forall a \in A \exists g \in P (f \subseteq g \text{ and } a \in \mathrm{dom}(g)) \tag{5.8}$$

$$\forall f \in P \forall b \in B \exists g \in P (f \subseteq g \text{ and } b \in \mathrm{rng}(g)). \tag{5.9}$$

The structures $\mathcal{A}$ and $\mathcal{B}$ are said to be *partially isomorphic*, in symbols $\mathcal{A} \simeq_p \mathcal{B}$, if there is a back-and-forth set for them.

Lemma 5.15 *The relation $\simeq_p$ is an equivalence relation on* $\mathrm{Str}(L)$.

Proof The relation $\simeq_p$ is reflexive, because $\{id_A\}$ is a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. If P is a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$, then $\{f^{-1} : f \in P\}$ is a back-and-forth set for $\mathcal{B}$ and $\mathcal{A}$. Finally, if P_1 is a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$ and P_2 is a back-and-forth set for $\mathcal{B}$ and $\mathcal{C}$, then $\{f_2 \circ f_1 : f_1 \in P_1, f_2 \in P_2\}$ is a back-and-forth set for $\mathcal{A}$ and $\mathcal{C}$, where we stipulate $\mathrm{dom}(f_2 \circ f_1) = f_1^{-1}(\mathrm{dom}(f_2))$. □

Proposition 5.16 *If $\mathcal{A} \simeq_p \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are countable, then $\mathcal{A} \cong \mathcal{B}$.*

Proof Let us enumerate A as $(a_n : n < \omega)$ and B as $(b_n : n < \omega)$. Let P be a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. Since $P \neq \emptyset$, there is some $f_0 \in P$. We define a sequence $(f_n : n < \omega)$ of elements of P as follows: Suppose $f_n \in P$ is defined. If n is even, say $n = 2m$, let $y \in B$ and $f_{n+1} \in P$ such that $f_n \cup \{(a_m, y)\} \subseteq f_{n+1}$. If n is odd, say $n = 2m + 1$, let $x \in A$ and $f_{n+1} \in P$ such that $f_n \cup \{(x, b_m)\} \subseteq f_{n+1}$. Finally, let

$$f = \bigcup_{n=0}^{\infty} f_n.$$

Clearly, $f : \mathcal{A} \cong \mathcal{B}$. □

This proposition is not true for uncountable structures. Indeed, let $L = \emptyset$ and let $\mathcal{A}$ and $\mathcal{B}$ be any infinite L-structures. Then there is a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$ (Exercise 5.28). Thus $\mathcal{A} \simeq_p \mathcal{B}$. But $\mathcal{A} \not\cong \mathcal{B}$ if, for example, $A = \mathbb{Q}$ and $B = \mathbb{R}$. The failure of Proposition 5.16 to generalize is a major topic in the sequel.

Proposition 5.17 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are dense linear orders without endpoints. Then $\mathcal{A} \simeq_p \mathcal{B}$.*

Proof Let $P = \{f \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : \mathrm{dom}(f) \text{ is finite}\}$. It turns out that this straightforward choice works. Clearly, $P \neq \emptyset$. Suppose then $f \in P$ and $a \in A$. Let us enumerate f as $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ where $a_1 < \ldots < a_n$. Since f is a partial isomorphism, also $b_1 < \ldots < b_n$. Now we consider different cases. If $a < a_1$, we choose $b < b_1$ and then $f \cup \{(a, b)\} \in P$. If $a_i < a < a_{i+1}$, we choose $b \in B$ so that $b_i < b < b_{i+1}$ and then $f \cup \{(a, b)\} \in P$. If $a_n < a$, we choose $b > b_n$ and again $f \cup \{(a, b)\} \in P$. Finally, if $a = a_i$, we let $b = b_i$ and then $f \cup \{(a, b)\} = f \in P$. We have proved (5.8). Condition (5.9) is proved similarly. □

Putting Proposition 5.16 and Proposition 5.17 together yields the famous result of Cantor (1895): countable dense linear orders without endpoints are isomorphic. See Exercise 6.29 for a more general result.

5.5 The Ehrenfeucht–Fraïssé Game

In Section 4.3 we introduced the Ehrenfeucht–Fraïssé Game played on two graphs. This game was used to measure to what extent two graphs have similar properties, especially properties expressible in the first-order language of graphs limited to a fixed quantifier rank. In this section we extend this game to the context of arbitrary structures, not just graphs.

Let us recall the basic idea behind the Ehrenfeucht–Fraïssé Game. Suppose $\mathcal{A}$ and $\mathcal{B}$ are L-structures for some relational L. We imagine a situation in which two mathematicians argue about whether $\mathcal{A}$ and $\mathcal{B}$ are isomorphic or not. The mathematician that we denote by **II** claims that they are isomorphic, while the other mathematician whom we call **I** claims the models have an intrinsic structural difference and they cannot possibly be isomorphic.

The matter would be quickly resolved if **II** was required to show the claimed isomorphism. But the rules of the game are different. The rules are such that **II** is required to show only small pieces of the claimed isomorphism.

More exactly, **I** asks what is the image of an element a_1 of A that he chooses

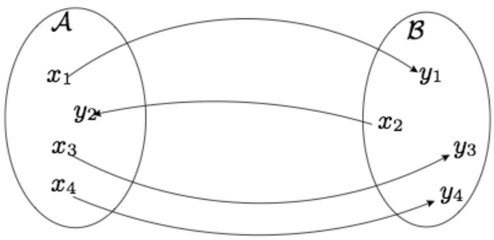


Figure 5.5 The Ehrenfeucht–Fraïssé Game.

at will. Then **II** is required to respond with some element b_1 of B so that

$$\{(a_1, b_1)\} \in \text{Part}(\mathcal{A}, \mathcal{B}). \tag{5.10}$$

Alternatively, **I** might have chosen an element b_1 of B and then **II** would have been required to produce an element a_1 of A such that (5.10) holds. The one-element mapping $\{(a_1, b_1)\}$ is called the *position* in the game after the first move.

Now the game goes on. Again **I** asks what is the image of an element a_2 of A (or alternatively he can ask what is the pre-image of an element b_2 of B). Then **II** produces an element b_2 of B (or in the alternative case an element a_2 of A). In either case the choice of **II** has to satisfy

$$\{(a_1, b_1), (a_2, b_2)\} \in \text{Part}(\mathcal{A}, \mathcal{B}). \tag{5.11}$$

Again, $\{(a_1, b_1), (a_2, b_2)\}$ is called the position after the second move.

We continue until the position

$$\{(a_1, b_1), \ldots, (a_n, b_n)\} \in \text{Part}(\mathcal{A}, \mathcal{B})$$

after the n^{th} move has been produced. If **II** has been able to play all the moves according to the rules she is declared the winner. Let us call this game $\text{EF}_n(\mathcal{A}, \mathcal{B})$. Figure 5.5 pictures the situation after four moves. If **II** can win repeatedly whatever moves **I** plays, we say that **II** has a *winning strategy*.

Example 5.18 Suppose $\mathcal{A}$ and $\mathcal{B}$ are two L-structures and $L = \emptyset$. Thus the structures $\mathcal{A}$ and $\mathcal{B}$ consist merely of a universe with no structure on it. In this singular case any one-to-one mapping is a partial isomorphism. The only thing player **II** has to worry about, say in (5.11), is that $a_1 = a_2$ if and only if $b_1 = b_2$. Thus **II** has a winning strategy in $\text{EF}_n(\mathcal{A}, \mathcal{B})$ if A and B both have at least n elements. So **II** can have a winning strategy even if A and B have different cardinality and there could be no isomorphism between them for the trivial reason that there is no bijection. The intuition here is that by playing a finite number of elements, or even $\aleph_0$ many, it is not possible to get hold of the cardinality of the universe if it is infinite.

Example 5.19 Let $\mathcal{A}$ be a linear order of length 3 and $\mathcal{B}$ a linear order of length 4. How many moves does **I** need to beat **II**? Suppose $A = \{a_1, a_2, a_3\}$ in increasing order and $B = \{b_1, b_2, b_3, b_4\}$ in increasing order. Clearly, if **I** plays at any point the smallest element, also **II** has to play the smallest element or face defeat on the next move. Also, if **I** plays at any point the smallest but one element, also **II** has to play the smallest but one element or face defeat in two moves. Now in $\mathcal{A}$ the smallest but one element is the same as the largest but one element, while in $\mathcal{B}$ they are different. So if **I** starts with a_2, **II** has to play b_2 or b_3, or else she loses in one move. Suppose she plays b_2. Now **I** plays b_3 and **II** has no good moves left. To obey the rules, she must play a_3. That is how long she can play, for now when **I** plays b_4, **II** cannot make a legal move anymore. In fact **II** has a winning strategy in $\mathrm{EF}_2(\mathcal{A}, \mathcal{B})$ but **I** has a winning strategy in $\mathrm{EF}_3(\mathcal{A}, \mathcal{B})$.

We now proceed to a more exact definition of the Ehrenfeucht–Fraïssé Game.

Definition 5.20 Suppose L is a vocabulary and $\mathcal{M}, \mathcal{M}'$ are L-structures such that $M \cap M' = \emptyset$. The *Ehrenfeucht–Fraïssé Game* $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$ is the game $\mathcal{G}_n(M \cup M', W_n(\mathcal{M}, \mathcal{M}'))$, where $W_n(\mathcal{M}, \mathcal{M}') \subseteq (M \cup M')^{2n}$ is the set of $p = (x_0, y_0, \dots, x_{n-1}, y_{n-1})$ such that:

(G1) For all $i < n$: $x_i \in M \iff y_i \in M'$.
(G2) If we denote

$$v_i = \begin{cases} x_i & \text{if } x_i \in M \\ y_i & \text{if } y_i \in M \end{cases} \quad v_i' = \begin{cases} x_i & \text{if } x_i \in M' \\ y_i & \text{if } y_i \in M', \end{cases}$$

then

$$f_p = \{(v_0, v_0'), \dots, (v_{n-1}, v_{n-1}')\}$$

is a partial isomorphism $\mathcal{M} \to \mathcal{M}'$.

We call v_i and v_i' *corresponding* elements. The infinite game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ is defined quite similarly, that is, it is the game $\mathcal{G}_\omega(M \cup M', W_\omega(\mathcal{M}, \mathcal{M}'))$, where $W_\omega(\mathcal{M}, \mathcal{M}')$ is the set of $p = (x_0, y_0, x_1, y_1, \dots)$ such that for all $n \in \mathbb{N}$ we have $(x_0, y_0, \dots, x_{n-1}, y_{n-1}) \in W_n(\mathcal{M}, \mathcal{M}')$.

Note that the game EF_ω is a closed game.

Proposition 5.21 *Suppose L is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are L-structures. The following are equivalent:*

1. *$\mathcal{A} \simeq_p \mathcal{B}$.*
2. ***II** has a winning strategy in* $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$.

Proof Assume $A \cap B = \emptyset$. Let P be first a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. We define a winning strategy $\tau = (\tau_i : i < \omega)$ for **II**. Since $P \neq \emptyset$ we can fix an element f of P. Condition (5.8) tells us that if $a_1 \in A$, then there are $b_1 \in B$ and g such that

$$f \cup \{(a_1, b_1)\} \subseteq g \in P. \tag{5.12}$$

Let $\tau_0(a_1)$ be one such b_1. Likewise, if $b_1 \in B$, then there are $a_1 \in A$ such that (5.12) holds and we can let $\tau_0(b_1)$ be some such a_1. We have defined $\tau_0(c_1)$ whatever c_1 is. To define $\tau_1(c_1, c_2)$, let us assume **I** played $c_1 = a_1 \in A$. Thus (5.12) holds with $b_1 = \tau_0(a_1)$. If $c_2 = a_2 \in A$ we can use (5.8) again to find $b_2 = \tau_1(a_1, a_2) \in B$ and h such that

$$f \cup \{(a_1, b_1), (a_2, b_2)\} \subseteq h \in P.$$

The pattern should now be clear. The back-and-forth set P guides **II** to always find a valid move. Let us then write the proof in more detail: Suppose we have defined τ_i for $i < j$ and we want to define τ_j. Suppose player **I** has played $x_0, \ldots, x_{j-1}$ and player **II** has followed τ_i during round $i < j$. During the inductive construction of τ_i we took care to define also a partial isomorphism $f_i \in P$ such that $\{v_0, \ldots, v_{i-1}\} \subseteq \mathrm{dom}(f_{i-1})$. Now player **I** plays x_j. By assumption there is $f_j \in P$ extending f_{j-1} such that if $x_j \in A$, then $x_j \in \mathrm{dom}(f_j)$ and if $x_j \in B$, then $x_j \in \mathrm{rng}(f_j)$. We let $\tau_j(x_0, \ldots, x_j) = f_j(x_j)$ if $x_j \in A$ and $\tau_j(x_0, \ldots, x_j) = f_j^{-1}(x_j)$ otherwise. This ends the construction of τ_j. This is a winning strategy because every f_p extends to a partial isomorphism $\mathcal{M} \to \mathcal{N}$.

For the converse, suppose $\tau = (\tau_n : n < \omega)$ is a winning strategy of **II**. Let Q consist of all plays of $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ in which player **II** has used τ. Let P consist of all possible f_p where p is a position in the game $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ with an extension in Q. It is clear that P is non-void and has the properties (5.8) and (5.9). □

To prove partial isomorphism of two structures we now have two alternative methods:

1. Construct a back-and-forth set.
2. Show that player **II** has a winning strategy in EF_ω.

By Proposition 5.21 these methods are equivalent. In practice one uses the game as a guide to intuition and then for a formal proof one usually uses a back-and-forth set.

5.6 Back-and-Forth Sequences

Back-and-forth sets and winning strategies of player **II** in the Ehrenfeucht–Fraïssé Game EF_ω correspond to each other. There is a more refined concept, called a back-and-forth sequence, which corresponds to a winning strategy of player **II** in the finite game EF_n.

Definition 5.22 A *back-and-forth sequence* $(P_i : i \leq n)$ is defined by the conditions

$$\emptyset \neq P_n \subseteq \ldots \subseteq P_0 \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B}). \tag{5.13}$$

$$\forall f \in P_{i+1} \forall a \in A \exists b \in B \exists g \in P_i (f \cup \{(a,b)\} \subseteq g) \text{ for } i < n. \tag{5.14}$$

$$\forall f \in P_{i+1} \forall b \in B \exists a \in A \exists g \in P_i (f \cup \{(a,b)\} \subseteq g) \text{ for } i < n. \tag{5.15}$$

If P is a back-and-forth set, we can get back-and-forth sequences $(P_i : i \leq n)$ of any length by choosing $P_i = P$ for all $i \leq n$. But the converse is not true: the sets P_i need by no means be themselves back-and-forth sets. Indeed, pairs of countable models may have long back-and-forth sequences without having any back-and-forth sets. Let us write

$$\mathcal{A} \simeq_p^n \mathcal{B}$$

if there is a back-and-forth sequence of length n for $\mathcal{A}$ and $\mathcal{B}$.

Lemma 5.23 *The relation $\simeq_p^n$ is an equivalence relation on $Str(L)$.*

Proof Exactly as Lemma 5.15. □

Example 5.24 We use $(\mathbb{N} + \mathbb{N}, <)$ to denote the linear order obtained by putting two copies of $(\mathbb{N}, <)$ one after the other. (The ordinal of this order is $\omega + \omega$.) Now $(\mathbb{N}, <) \simeq_p^2 (\mathbb{N} + \mathbb{N}, <)$, for we may take

$$P_2 = \{\emptyset\}.$$
$$P_1 = \{\{(a,b)\} : 0 < a \in \mathbb{N},\ 0 < b \in \mathbb{N} + \mathbb{N}\} \cup \{(0,0)\} \cup P_2.$$
$$P_0 = \{\{(a_0,b_0),(a_1,b_1)\} : a_0 < a_1 \in \mathbb{N},\ b_0 < b_1 \in \mathbb{N} + \mathbb{N}\} \cup P_1.$$

Note that $(\mathbb{N}, <) \not\simeq_p^3 (\mathbb{N} + \mathbb{N}, <)$.

Proposition 5.25 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are discrete linear orders (i.e. every element with a successor has an immediate successor and every element with a predecessor has an immediate predecessor) with no endpoints, and $n \in \mathbb{N}$. Then $\mathcal{A} \simeq_p^n \mathcal{B}$.*

Proof Let P_i consist of $f \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$ with the following property: $f = \{(a_0, b_0), \ldots, (a_{n-i-1}, b_{n-i-1})\}$ where

$$a_0 \leq \ldots \leq a_{n-i-1},$$
$$b_0 \leq \ldots \leq b_{n-i-1},$$

and for all $0 \leq j < n - i - 1$ if $|(a_j, a_{j+1})| < 2^i$ or $|(b_j, b_{j+1})| < 2^i$, then $|(a_j, a_{j+1})| = |(b_j, b_{j+1})|$. □

Example 5.26 $(\mathbb{Z}, <) \simeq_p^n (\mathbb{Z} + \mathbb{Z}, <)$ for all $n \in \mathbb{N}$, but note that $(\mathbb{Z}, <) \not\simeq_p (\mathbb{Z} + \mathbb{Z}, <)$.

Proposition 5.27 *Suppose L is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are L-structures. The following are equivalent:*

1. $\mathcal{A} \simeq_p^n \mathcal{B}$.
2. **II** *has a winning strategy in* $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$.

Proof Let us assume $A \cap B = \emptyset$. Let $(P_i : i \leq n)$ be a back-and-forth sequence for $\mathcal{A}$ and $\mathcal{B}$. We define a winning strategy $\tau = (\tau_i : i \leq n)$ for **II**. Since $P_n \neq \emptyset$ we can fix an element f of P_n. Condition (5.14) tells us that if $a_1 \in A$, then there are $b_1 \in B$ and g such that

$$f \cup \{(a_1, b_1)\} \subseteq g \in P_{n-1}. \tag{5.16}$$

Let $\tau_0(a_1)$ be one such b_1. Likewise, if $b_1 \in B$, then there are $a_1 \in A$ such that (5.16) holds and we can let $\tau_0(b_1)$ be some such a_1. We have defined $\tau_0(c_1)$ whatever c_1 is. To define $\tau_1(c_1, c_2)$, let us assume **I** played $c_1 = a_1 \in A$. Thus (5.16) holds with $b_1 = \tau_0(a_1)$. If $c_2 = a_2 \in A$ we can use (5.13) again to find $b_2 = \tau_1(a_1, a_2) \in B$ and h such that

$$f \cup \{(a_1, b_1), (a_2, b_2)\} \subseteq h \in P_{n-2}.$$

The pattern should be clear now. As before, the back-and-forth sequence guides **II** to always find a valid move. Let us then write the proof in more detail: Suppose we have defined τ_i for $i < j$ and we want to define τ_j. Suppose player **I** has played $x_0, \ldots, x_{j-1}$ and player **II** has followed τ_i during round $i < j$. During the inductive construction of τ_i we took care to define also a partial isomorphism $f_i \in P_{n-i}$ such that $\{v_0, \ldots, v_{i-1}\} \subseteq \mathrm{dom}(f_i)$. Now player **I** plays x_j. By assumption there is $f_j \in P_{n-j}$ extending f_{j-1} such that if $x_j \in A$, then $x_j \in \mathrm{dom}(f_j)$ and if $x_j \in B$, then $x_j \in \mathrm{rng}(f_j)$. We let $\tau_j(x_0, \ldots, x_j) = f_j(x_j)$ if $x_j \in A$ and $\tau_j(x_0, \ldots, x_j) = f_j^{-1}(x_j)$ otherwise. This ends the construction of τ_j. This is a winning strategy because every f_p extends to a partial isomorphism $\mathcal{M} \to \mathcal{N}$.

For the converse, suppose $\tau = (\tau_i : i \le n)$ is a winning strategy of **II**. Let Q consist of all plays of $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ in which player **II** has used τ. Let P_{n-i} consist of all possible f_p where $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ is a position in the game $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ with an extension in Q. It is clear that $(P_i : i \le n)$ has the properties (5.13) and (5.14). Note that:

$$P_n = \{\emptyset\}$$

$$P_{n-1} = \{(x_0, \tau_0(x_0)) : x_0 \in A \cup B\}$$

$$P_{n-2} = \{(x_0, \tau_0(x_0), x_1, \tau_1(x_0, x_1)) : x_0, x_1 \in A \cup B\}$$

$$P_0 = \{(x_0, \tau_0(x_0), \ldots, x_{n-1}, \tau_{n-1}(x_0, \ldots, x_{n-1})) : x_0, \ldots, x_{n-1} \in A \cup B\}.$$

□

5.7 Historical Remarks and References

Back-and-forth sets are due to Fraïssé (1955). The Ehrenfeucht–Fraïssé Game was introduced in Ehrenfeucht (1957) and Ehrenfeucht (1960/1961). Back-and-forth sequences were introduced in Karp (1965). Exercise 5.40 is from Ellentuck (1976). Exercise 5.40 is from Ellentuck (1976). Exercise 5.54 is from Barwise (1975). Exercise 5.71 is from Rosenstein (1982).

Exercises

5.1 Show that isomorphism of structures is an equivalence relation in the sense that it is reflexive, symmetric, and transitive.

5.2 Suppose L is a finite vocabulary, $\mathcal{B}$ is a countable L-model, and $\{b_n : n < \omega\}$ is an enumeration of the domain B of $\mathcal{B}$. Suppose $\mathcal{A}$ is a countable L-model. Show that the following are equivalent:

(1) $\mathcal{A} \cong \mathcal{B}$.

(2) There is an enumeration $\{a_n : n < \omega\}$ of the domain of $\mathcal{A}$ so that for all atomic L-formulas $\theta(x_0, \ldots, x_n)$ and all $n < \omega$ we have

$$\mathcal{A} \models \theta(a_0, \ldots, a_n) \iff \mathcal{B} \models \theta(b_0, \ldots, b_n).$$

5.3 Suppose L is a vocabulary and $\mathcal{M}$ is an L-structure. Show that the set $\mathrm{Aut}(\mathcal{M})$ of automorphisms of $\mathcal{M}$ forms a group under the operation of composition of functions.

5.4 Give an example of $\mathcal{M}$ such that $\mathrm{Aut}(\mathcal{M})$ (see the previous exercise) is:

1. The trivial one-element group.
2. A non-trivial abelian group (e.g. the additive group of the integers).
3. A non-abelian group (e.g. the symmetric group S_3).

5.5 How many automorphisms do the following structures have.

1. A linear order of n elements.
2. $(\mathbb{N}, <)$.
3. $(\mathbb{Z}, <)$.
4. $(\mathbb{Q}, <)$.

5.6 Show that if $\mathcal{A}$ and $\mathcal{B}$ are unary structures, then $\mathcal{A} \cong \mathcal{B}$ if and only if for all $\epsilon : \{1, \ldots, n\} \to \{0, 1\}$ we have $|C_\epsilon(\mathcal{A})| = |C_\epsilon(\mathcal{B})|$. Easier version: Show that if $\mathcal{A}$ and $\mathcal{B}$ are unary structures with a finite universe of size n, then $\mathcal{A} \cong \mathcal{B}$ if and only if for all $\epsilon : \{1, \ldots, n\} \to \{0, 1\}$ we have $|C_\epsilon(\mathcal{A})| = |C_\epsilon(\mathcal{B})|$.

5.7 Suppose $\mathcal{M}$ is a unary structure in which every ϵ-constituent has exactly three elements. How many elements does $\mathcal{M}$ have? How many automorphisms does $\mathcal{M}$ have?

5.8 $L = \{P_1, \ldots, P_m\}$, where each P_i is unary. Show that the number of non-isomorphic L-structures on the universe $\{1, \ldots, n\}$ is $\binom{n+2^m-1}{2^m-1}$.

5.9 Describe the group of automorphisms of a finite unary structure.

5.10 Suppose $\mathcal{M}$ is an equivalence relation with a finite universe such that $EC_n(\mathcal{M}) = 2$ for each $n = 1, \ldots, 5$ and $EC_n(\mathcal{M}) = 0$ for other n. How many elements are there in the universe of $\mathcal{M}$? How many automorphisms does $\mathcal{M}$ have?

5.11 Show that for any $m \in \mathbb{N}$ there is $m^* \in \mathbb{N}$ such that if $n \geq m^*$ then there are more than n^m non-isomorphic equivalence relations on the universe $\{1, \ldots, n\}$. Conclude that for any $m \in \mathbb{N}$ there is $m^* \in \mathbb{N}$ such that if $n \geq m^*$ then there are more non-isomorphic equivalence relations on the universe $\{1, \ldots, n\}$ than non-isomorphic $\{P_1, \ldots, P_m\}$-structures, where each P_i is unary.

5.12 Show that if $\mathcal{A}$ and $\mathcal{B}$ are equivalence relations, then $\mathcal{A} \cong \mathcal{B}$ if and only if for all $\kappa \leq |A \cup B|$ we have $EC_\kappa(\mathcal{A}) = EC_\kappa(\mathcal{B})$. Easier version: Show that if $\mathcal{A}$ and $\mathcal{B}$ are equivalence relations with a finite universe of size n, then $\mathcal{A} \cong \mathcal{B}$ if and only if for all $m \leq n$ we have $EC_m(\mathcal{A}) = EC_m(\mathcal{B})$.

5.13 Describe the group of automorphisms of a finite equivalence relation.

5.14 Show that if $\mathcal{M}$ and $\mathcal{N}$ are countable dense linear orders, then $\mathcal{M} \cong \mathcal{N}$ if and only if $SG(\mathcal{M}) = SG(\mathcal{N})$. Demonstrate that this is not true for non-dense countable linear orders or for uncountable dense linear orders.

5.15 Show that two well-orders $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if $o(\mathcal{M}) = o(\mathcal{N})$.

5.16 Prove that two well-founded trees $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if $\mathrm{stp}_\mathcal{M} = \mathrm{stp}_\mathcal{N}$.

5.17 Prove that two successor structures $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if $CC_a(\mathcal{M}) = CC_a(\mathcal{N})$ for all $a \in \mathbb{N} \cup \{\infty\}$. Easier version: Prove that two successor structures $\mathcal{M}$ and $\mathcal{N}$ both of which have only finitely many components are isomorphic if and only if $CC_a(\mathcal{M}) = CC_a(\mathcal{N})$ for all $a \in \mathbb{N} \cup \{\infty\}$.

5.18 Show that any uncountable collection of countable non-isomorphic successor structures has to contain a successor structure with infinitely many cycle components.

5.19 Describe the group of automorphisms of a successor structure with n $\mathbb{Z}$-components and m_i i-cycle components for $i = 1, \ldots, k$.

5.20 Give an example of an infinite structure $\mathcal{M}$ with no substructures $\mathcal{N} \neq \mathcal{M}$.

5.21 Consider $\mathcal{M} = (\mathbb{Z}, +)$. What is $[X]_\mathcal{M}$, if X is

1. $\{0\}$,
2. $\{1\}$,
3. $\{2, -2\}$.

5.22 Consider $\mathcal{M} = (\mathbb{Z}, +, -)$. What is $[X]_\mathcal{M}$, if X is $\{13, 17\}$?

5.23 Suppose $\mathcal{M}$ is a successor structure consisting of the standard component and two five-cycles. Show that there are exactly four possibilities for the set $[X]_\mathcal{M}$.

5.24 Show that the universe of $[X]_\mathcal{M}$ is the intersection of all universes of substructures $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq N$.

5.25 Prove Lemma 5.12.

5.26 Show that every Boolean algebra $\mathcal{M}$ is isomorphic to a substructure of $(\mathcal{P}(A), \subseteq)$, where A is the set of all ultrafilters of $\mathcal{M}$. (This is the so-called *Stone's Representation Theorem*.)

5.27 Show that every tree every element of which has height $< \omega$ is isomorphic to a substructure of the tree $(A^{<\omega}, \leq)$ for some set A.

5.28 Suppose $L = \emptyset$. Show that any two infinite L-structures are partially isomorphic.

5.29 Suppose $L = \{P_1, \ldots, P_n\}$ is a *unary* vocabulary. Suppose we have two L-structures $\mathcal{M}$ and $\mathcal{N}$ satisfying the following condition: For all $\epsilon : \{1, \ldots, n\} \to \{0, 1\}$ and all $m \in \mathbb{N}$ it holds that

$$|C_\epsilon(\mathcal{M})| = m \iff |C_\epsilon(\mathcal{N})| = m.$$

Show that this is a necessary and sufficient condition for the two structures to be partially isomorphic.

5.30 Suppose that two equivalence relations $\mathcal{M}$ and $\mathcal{N}$ satisfy the following conditions for all $n, m < \omega$:

1. $EC_n(\mathcal{M}) = m \iff EC_n(\mathcal{N}) = m.$
2. If one has exactly m infinite classes, then so does the other. In symbols:

$$\sum_{\aleph_0 \leq \kappa \leq |M|} EC_\kappa(\mathcal{M}) = m \iff \sum_{\aleph_0 \leq \kappa \leq |N|} EC_\kappa(\mathcal{N}) = m.$$

Show that these are a necessary and sufficient condition for the two structures to be partially isomorphic.

5.31 For elements t of a well-founded tree $\mathcal{M}$ we can define

$$\mathrm{dom}(\mathrm{stp}'_{\mathcal{M},t}) = \{\mathrm{stp}'_{\mathcal{M},s} : s \in \mathrm{ImSuc}(t)\}$$

$$\mathrm{stp}'_{\mathcal{M},t}(\mathrm{stp}'_{\mathcal{M},s}) = \min(\aleph_0, |\{s' \in \mathrm{ImSuc}(t) : \mathrm{stp}'_{\mathcal{M},s} = \mathrm{stp}'_{\mathcal{M},s'}\}|).$$

Suppose $\mathcal{M}$ and $\mathcal{N}$ are well-founded trees such that $\mathrm{stp}'_{\mathcal{M}} = \mathrm{stp}'_{\mathcal{N}}$. Show that $\mathcal{M}$ and $\mathcal{N}$ are partially isomorphic. Give an example of two well-founded partially isomorphic trees that are not isomorphic.

5.32 Suppose that $\mathcal{M}$ and $\mathcal{N}$ are successor structures, $f \in \mathrm{Part}(\mathcal{M}, \mathcal{N})$. Show:

1. f maps elements of the standard component of $\mathcal{M}$ to elements of the standard component of $\mathcal{N}$.
2. f maps elements of a cycle component of $\mathcal{M}$ of size n to elements of a cycle component of $\mathcal{N}$ of size n.
3. f maps elements of a $\mathbb{Z}$-component of $\mathcal{M}$ to elements of a $\mathbb{Z}$-component of $\mathcal{N}$.

5.33 Suppose that two successor structures $\mathcal{M}$ and $\mathcal{N}$ satisfy the following conditions for all $n, m < \omega$:

1. $CC_n(\mathcal{M}) = m \iff CC_n(\mathcal{N}) = m$.
2. $CC_\infty(\mathcal{M}) = m \iff CC_\infty(\mathcal{N}) = m$.

Show that the successor structures are partially isomorphic.

5.34 Show that $\mathrm{Part}(\mathcal{M}, \mathcal{N})$ is closed under unions of chains, i.e. if $f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$ are in $\mathrm{Part}(\mathcal{M}, \mathcal{N})$, then so is $\bigcup_{n=0}^{\infty} f_n$.

5.35 Suppose $(\mathbb{R}, <, f) \simeq_p (\mathbb{R}, <, g))$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that g is also continuous.

5.36 If (M, d), $d : M \times M \to \mathbb{R}$, is a metric space, we can think of (M, d) as a an L-structure $\mathcal{M} = (M, d, \mathbb{R}, <_{\mathbb{R}})$, where L contains a binary function symbol, a unary predicate symbol, and a binary relation symbol. Show that there are a separable metric space $\mathcal{M} = (M, d, \mathbb{R}, <_{\mathbb{R}})$ and a non-separable metric space $\mathcal{M}' = (M', d', \mathbb{R}, <_{\mathbb{R}})$ such that $\mathcal{M} \simeq_p \mathcal{M}'$.

5.37 Show that there is a complete separable metric space (Polish space) $\mathcal{M} = (M, d, \mathbb{R}, <_{\mathbb{R}})$ and a non-complete separable metric space $\mathcal{M}' = (M', d', \mathbb{R}, <_{\mathbb{R}})$ such that $\mathcal{M} \simeq_p \mathcal{M}'$.

5.38 Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures of the same relational vocabulary L and $A \cap B = \emptyset$. The *disjoint sum* of $\mathcal{A}$ and $\mathcal{B}$ is the L-structure

$$(A \cup B, (R^{\mathcal{A}} \cup R^{\mathcal{B}})_{R \in L}).$$

Show that partial isomorphism is preserved by disjoint sums of models.

5.39 Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures of the same vocabulary L. The *direct product* of $\mathcal{A}$ and $\mathcal{B}$ is the L-structure

$$(A \times B, (R^{\mathcal{A}} \times R^{\mathcal{B}})_{R \in L},$$

$$(((a_0, b_0) \ldots, (a_n, b_n)) \mapsto (f^{\mathcal{A}}(a_0, \ldots, a_n), f^{\mathcal{B}}(b_0, \ldots, b_n)))_{f \in L},$$

$$((c^{\mathcal{A}}, c^{\mathcal{B}}))_{c \in L}).$$

Show that partial isomorphism is preserved by direct products of models.

5.40 Show that if two structures are partially isomorphic, then they are *potentially isomorphic,*[2] i.e. there is a forcing extension in which they are isomorphic. Conversely, show that if two structures are potentially isomorphic, then they are partially isomorphic.

5.41 Consider $\mathrm{EF}_2(\mathcal{M}, \mathcal{N})$, where $\mathcal{M} = (\mathbb{R} \times \{0\}, f)$, $f(x, 0) = (x^2, 0)$ and $\mathcal{N} = (\mathbb{R} \times \{1\}, g)$, $g(x, 1) = (x^3, 1)$. Player **I** can win even without looking at the moves of **II**. How?

5.42 Consider $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{N})$, where $\mathcal{M} = (\mathbb{R} \times \{0\}, f)$, $f(x, 0) = (x^3, 0)$ and $\mathcal{N} = (\mathbb{R} \times \{1\}, g)$, $g(x, 1) = (x^5, 1)$. After a few moves player **I** resigns. Can you explain why?

[2] Some authors use the term potential isomorphism for partial isomorphism.

5.43 Consider $\mathrm{EF}_2(\mathcal{M},\mathcal{N})$, where $\mathcal{M} = (\mathbb{Z}, \{(a,b) : a-b = 10\})$ and $\mathcal{N} = (\mathbb{Q}, \{(a,b) : a-b = 2/3\})$. Suppose we are in position $(-8, -1/4)$ (i.e. $x_0 = -8$ and $y_0 = -1/4$). Then **I** plays $x_1 = 11/12$. What would be a good move for **II**?

5.44 Consider $\mathrm{EF}_\omega(\mathcal{M},\mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are as in the previous exercise. Player **I** resigns before the game even starts. Can you explain why?

5.45 Suppose M and N are disjoint sets with 10 elements each. Let $c \in M$ and $d \in N$. Who has a winning strategy in $\mathrm{EF}_\omega(\mathcal{M},\mathcal{N})$ in the following cases:

1. $\mathcal{M} = (M, \{(a,b,c) : a = b\}), \mathcal{N} = (N, \{(a,b,d) : a = b\})$,
2. $\mathcal{M} = (M, \{(a,b,e) : a = b\}), \mathcal{N} = (N, \{(a,b,e) : b = e\})$.

5.46 Who has a winning strategy in $\mathrm{EF}_\omega(\mathcal{M},\mathcal{N})$ in the following cases:

1. $\mathcal{M} = (\mathbb{Q}, <, 1855), \mathcal{N} = (\mathbb{R}, <, 1854)$,
2. $\mathcal{M} = (\mathbb{N}, <, 1855), \mathcal{N} = (\mathbb{N}, <, 1854)$.

5.47 Show that $(\mathcal{P}(X), \subseteq) \simeq_p (\mathcal{P}(Y), \subseteq)$, if X and Y are disjoint infinite sets. (Hint: Consider the set of finite partial isomorphisms of the form $\{(A_0, B_0), \ldots, (A_{i-1}, B_{i-1})\}$, such that $(X, A_0, \ldots, A_{i-1})$ and $(Y, B_0, \ldots, B_{i-1})$ are partially isomorphic, and then use Exercise 5.29 of Section 5.4.)

5.48 Show that player **II** has a winning strategy in the game $\mathrm{EF}_\omega(\mathcal{M},\mathcal{N})$ for any two atomless (i.e. if $0 < x$ then there is y with $0 < y < x$) Boolean algebras $\mathcal{M}$ and $\mathcal{N}$.

5.49 Show that player **I** has a winning strategy in $\mathrm{EF}_2((\mathbb{Q}, +, -), (\mathbb{R}, +, -))$.

5.50 Consider $\mathrm{EF}_\omega((\mathbb{R}, +, -), (\mathbb{R} \times \mathbb{R}, +, -))$, where addition and substraction in $\mathbb{R} \times \mathbb{R}$ are defined componentwise. Show that player **II** has a winning strategy.

5.51 Show that partially isomorphic linear orders are isomorphic, if one is a well-order.

5.52 Show that infinite partially isomorphic structures have countably infinite isomorphic substructures.

5.53 Show that if one of two partially isomorphic trees is well-founded, then both are and the trees have the same rank. (Hint: For the second claim, prove first that if $\mathcal{M}$ is a well-founded tree, $t \in M$ and $\alpha < \mathrm{rk}_{\mathcal{M}}(t)$, then there is $t' \in M$ such that $\alpha = \mathrm{rk}_{\mathcal{M}}(t')$ and $t <^{\mathcal{M}} t'$.)

5.54 Suppose T is an axiomatization of set theory, at least as strong as the Kripke–Platek set theory KP (see Barwise (1975)). We say that a formula $\varphi(x_1, \ldots, x_n)$ of the language of set theory is *absolute* relative to T if for all transitive models M and N of T and for all $a_1, \ldots, a_n \in M$ we have

$$M \models \varphi(a_1, \ldots, a_n) \iff M' \models \varphi(a_1, \ldots, a_n).$$

Show that "x is a vocabulary, y and z are x-structures, and $y \simeq_p z$" can be defined with a formula $\varphi(x, y, z)$ which is absolute relative to T.

5.55 Suppose $\mathcal{A}$ is a linear order of length three and $\mathcal{B}$ a linear order of length four. Give a back-and-forth sequence of length two for $\mathcal{A}$ and $\mathcal{B}$.

5.56 Suppose $\mathcal{A}$ is a cycle of four vertices and $\mathcal{B}$ a cycle of five vertices. Give a back-and-forth sequence of length two for $\mathcal{A}$ and $\mathcal{B}$.

5.57 Suppose $\mathcal{A}$ is an equivalence relation of four classes each of size 3 and $\mathcal{B}$ an equivalence relation of three classes each of size 4. Give a back-and-forth sequence of length three for $\mathcal{A}$ and $\mathcal{B}$.

5.58 Suppose $\mathcal{A}$ is an equivalence relation of four classes each of size 2 and $\mathcal{B}$ an equivalence relation of three classes each of size 2. Give a back-and-forth sequence of length three for $\mathcal{A}$ and $\mathcal{B}$.

5.59 Suppose $\mathcal{A}$ is an equivalence relation of four classes each of size 2 plus one class of size 3, and $\mathcal{B}$ an equivalence relation of three classes each of size 2 plus one class of size 4. Give a back-and-forth sequence of length three for $\mathcal{A}$ and $\mathcal{B}$.

5.60 Suppose $\mathcal{A}$ and $\mathcal{B}$ are successor structures, both consisting of the standard component plus some cycle components. Suppose $\mathcal{A}$ has three five-cycles and $\mathcal{B}$ has four five-cycles. Give a back-and-forth sequence of length three for $\mathcal{A}$ and $\mathcal{B}$.

5.61 Show that $(7, <) \simeq_p^3 (8, <)$.

5.62 Show that $(\mathbb{Z}, <) \not\simeq_p^3 (\mathbb{Q}, <)$.

5.63 Show that $(\mathbb{N}, <) \not\simeq_p^3 (\mathbb{N} + \mathbb{N}, <)$.

5.64 Show that $(\mathbb{Z}, <) \not\simeq_p (\mathbb{Z} + \mathbb{Z}, <)$.

5.65 Show that $(\mathbb{N} + \mathbb{N}, <) \simeq_p^3 (\mathbb{N} + \mathbb{N} + \mathbb{N}, <)$

5.66 Finish the proof of Proposition 5.25.

5.67 Prove the claim of Example 5.26.

5.68 Let the game $\mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$ be like the game $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ except that **I** has to play $x_{2n} \in A$ and $x_{2n+1} \in B$ for all $n \in \mathbb{N}$. Show that player **II** has a winning strategy in $\mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$ if and only if she has a winning strategy in $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$.

5.69 Suppose $B = \{b_n : n \in \mathbb{N}\}$. Let the game $\mathrm{EF}^{**}_\omega(\mathcal{A}, \mathcal{B})$ be like the game $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ except that **I** has to play $x_{2n} \in A$ and $x_{2n+1} = b_n$ for all $n \in \mathbb{N}$. Show that player **II** has a winning strategy in $\mathrm{EF}^{**}_\omega(\mathcal{A}, \mathcal{B})$ if and only if she has a winning strategy in $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$.

5.70 Suppose $\mathcal{A}_0 = (A_0, <_0)$ and $\mathcal{A}_1 = (A_1, <_1)$ are linearly ordered sets. Show that if player **II** has a winning strategy both in $\mathrm{EF}_n(\mathcal{A}_0, \mathcal{B}_0)$ and in $\mathrm{EF}_n(\mathcal{A}_1, \mathcal{B}_1)$, then she has one in $\mathrm{EF}_n(\mathcal{A}_0 + \mathcal{A}_1, \mathcal{B}_0 + \mathcal{B}_1)$.

5.71 If $\mathcal{A} = (A, <)$ is a linearly ordered set and $a \in A$, then $\mathcal{A}^{<a}$ is the substructure of $\mathcal{A}$ generated by the set $\{x \in A : x < a\}$. Thus $\mathcal{A}^{<a}$ is the initial segment of $\mathcal{A}$ determined by a. Likewise, $\mathcal{A}^{>a}$ is the substructure of $\mathcal{A}$ generated by the set $\{x \in A : x > a\}$. Thus $\mathcal{A}^{>a}$ is the final segment of $\mathcal{A}$ determined by a. Show that if $\mathcal{A}$ and $\mathcal{B}$ are ordered sets, then player **II** has a winning strategy in $\mathrm{EF}_{n+1}(\mathcal{A}, \mathcal{B})$ if and only if

1. For every $a \in A$ there is $b \in B$ such that player **II** has a winning strategy in $\mathrm{EF}_n(\mathcal{A}^{<a}, \mathcal{B}^{<b})$ and in $\mathrm{EF}_n(\mathcal{A}^{>a}, \mathcal{B}^{>b})$.
2. For every $b \in B$ there is $a \in A$ such that player **II** has a winning strategy in $\mathrm{EF}_n(\mathcal{A}^{<a}, \mathcal{B}^{<b})$ and in $\mathrm{EF}_n(\mathcal{A}^{>a}, \mathcal{B}^{>b})$.

5.72 Suppose $n > 0$. Show that player **II** has a winning strategy in $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are linear orders with at least $2^n - 1$ elements.

5.73 Suppose $n > 0$. Show that player **I** has a winning strategy in $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are linear orders such that $\mathcal{A}$ has at least $2^n - 1$ elements and $\mathcal{B}$ has fewer than $2^n - 1$ elements.

5.74 Show that player **II** has a winning strategy in $\mathrm{EF}_n((\mathbb{N}, <), (\mathbb{N} + \mathbb{Z}, <))$ for every $n \in \mathbb{N}$.

5.75 An ordered set is scattered if it contains no substructure isomorphic to $(\mathbb{Q}, <)$. Show that if $\mathcal{M} \simeq_p \mathcal{N}$, where $\mathcal{N}$ is scattered, then $\mathcal{M}$ is scattered.

5.76 Suppose $\mathcal{T}$ is the tree of finite increasing sequences of rationals, and $\mathcal{T}'$ is the tree of finite increasing sequences of reals. Prove $\mathcal{T} \simeq_p T'$.

5.77 Suppose $\mathcal{T}$ is the tree of finite sequences of rationals, and $\mathcal{T}'$ is the tree of finite sequences of reals. Prove $\mathcal{T} \simeq_p T'$.

5.78 Suppose $\mathcal{T}$ is the tree of increasing sequences of length $\leq n$ of rationals, and $\mathcal{T}'$ is the tree of increasing sequences of length $\leq n$ of reals. Prove $\mathcal{T} \simeq_p T'$.

5.79 Suppose $\mathcal{T}$ is the tree of sequences of length $\leq n$ of rationals, and $\mathcal{T}'$ is the tree of sequences of length $\leq n$ of reals. Prove $\mathcal{T} \simeq_p T'$.

5.80 Suppose $\mathcal{T}$ is the tree of sequences of length $\leq n$ of elements of the set $\{1, \ldots, m\}$, and $\mathcal{T}'$ is the tree of sequences of length $\leq n$ of elements of $\{1, \ldots, m+1\}$. Prove $\mathcal{T} \simeq_p^m T'$.

6

First-Order Logic

6.1 Introduction

We have already discussed the *first-order language of graphs*. We now define the basic concepts of a more general first-order language, denoted FO, one which applies to any vocabulary, not just the vocabulary of graphs. First-order logic fits the Strategic Balance of Logic better than any other logic. It is arguably the most important of all logics. It has enough power to express interesting and important concept and facts, and still it is weak and flexible enough to permit powerful constructions as demonstrated, e.g. by the Model Existence Theorem below.

6.2 Basic Concepts

Suppose L is a vocabulary. The *logical symbols* of the first-order language (or logic) of the vocabulary L are $\approx, \neg, \wedge, \vee, \forall, \exists, (,), x_0, x_1, \ldots$. *Terms* are defined as follows: Constant symbols $c \in L$ are L-terms. Variables $x_0, x_1, \ldots$ are L-terms. If $f \in L$, $\#(f) = n$, and $t_1, \ldots, t_n$ are L-terms, then so is $ft_1 \ldots t_n$. *L-equations* are of the form $\approx tt'$ where t and t' are L-terms. *L-atomic formulas* are either L-equations or of the form $Rt_1 \ldots t_n$, where $R \in L$, $\#(R) = n$ and $t_1, \ldots, t_n$ are L-terms. A *basic formula* is an atomic formula or the negation of an atomic formula. *L-formulas* are of the form

$$\approx tt'$$
$$Rt_1 \ldots t_n$$
$$\neg\varphi$$
$$(\varphi \wedge \psi), (\varphi \vee \psi)$$
$$\forall x_n \varphi, \exists x_n \varphi$$

where $t, t', t_1, \ldots, t_n$ are L-terms, $R \in L$ with $\#(R) = n$, and φ and ψ are L-formulas.

Definition 6.1 An *assignment* for a set M is any function s with $\mathrm{dom}(s)$ a set of variables and $\mathrm{rng}(s) \subseteq \mathcal{M}$. The *value* $t^{\mathcal{M}}(s)$ of an L-term t in $\mathcal{M}$ under the assignment s is defined as follows: $c^{\mathcal{M}}(s) = \mathrm{Val}_{\mathcal{M}}(c)$, $x_n^{\mathcal{M}}(s) = s(x_n)$ and $(ft_1 \ldots t_n)^{\mathcal{M}}(s) = \mathrm{Val}_{\mathcal{M}}(f)(t_1^{\mathcal{M}}(s), \ldots, t_n^{\mathcal{M}}(s))$. The *truth* of L-formulas in $\mathcal{M}$ under s is defined as follows:

$$
\begin{array}{lll}
\mathcal{M} \models_s Rt_1 \ldots t_n & \text{iff} & (t_1^{\mathcal{M}}(s), \ldots, t_n^{\mathcal{M}}(s)) \in \mathrm{Val}_{\mathcal{M}}(R) \\
\mathcal{M} \models_s \approx t_1 t_2 & \text{iff} & t_1^{\mathcal{M}}(s) = t_2^{\mathcal{M}}(s) \\
\mathcal{M} \models_s \neg\varphi & \text{iff} & \mathcal{M} \not\models_s \varphi \\
\mathcal{M} \models_s (\varphi \wedge \psi) & \text{iff} & \mathcal{M} \models_s \varphi \text{ and } \mathcal{M} \models_s \psi \\
\mathcal{M} \models_s (\varphi \vee \psi) & \text{iff} & \mathcal{M} \models_s \varphi \text{ or } \mathcal{M} \models_s \psi \\
\mathcal{M} \models_s \forall x_n \varphi & \text{iff} & \mathcal{M} \models_{s[a/x_n]} \varphi \text{ for all } a \in \mathcal{M} \\
\mathcal{M} \models_s \exists x_n \varphi & \text{iff} & \mathcal{M} \models_{s[a/x_n]} \varphi \text{ for some } a \in \mathcal{M}, \\
& & \text{where } s[a/x_n](y) = \begin{cases} a & \text{if } y = x_n \\ s(y) & \text{otherwise.} \end{cases}
\end{array}
$$

We assume the reader is familiar with such basic concepts as free variable, sentence, substitution of terms for variables, etc. A standard property of first-order (or any other) logic is that $\mathcal{M} \models_s \varphi$ depends only on $\mathcal{M}$ and the values of s on the variables that are free in φ. A *sentence* is a formula φ without free variables. Then $\mathcal{M} \models \varphi$ means $\mathcal{M} \models_\emptyset \varphi$. In this case we say that φ is *true* in $\mathcal{M}$.

Convention: If φ is an L-formula with the free variables $x_1, \ldots, x_n$, we indicate this by writing φ as $\varphi(x_1, \ldots, x_n)$. If $\mathcal{M}$ is an L-structure and s is an assignment for M such that $\mathcal{M} \models_s \varphi$, we write $\mathcal{M} \models \varphi(a_1, \ldots, a_n)$, where $a_i = s(x_i)$ for $i = 1, \ldots, n$.

Definition 6.2 The *quantifier rank* of a formula φ, denoted $\mathrm{QR}(\varphi)$, is defined as follows: $\mathrm{QR}(\approx tt') = \mathrm{QR}(Rt_1 \ldots t_n) = 0$, $\mathrm{QR}(\neg\varphi) = \mathrm{QR}(\varphi)$, $\mathrm{QR}((\varphi \wedge \psi)) = \mathrm{QR}((\varphi \vee \psi)) = \max\{\mathrm{QR}(\varphi), \mathrm{QR}(\psi)\}$, $\mathrm{QR}(\exists x\varphi) = \mathrm{QR}(\forall x\varphi) = \mathrm{QR}(\varphi) + 1$. A formula φ is *quantifier free* if $\mathrm{QR}(\varphi) = 0$.

The quantifier rank is a measure of the longest sequence of "nested" quantifiers. In the first three of the following formulas the quantifiers $\forall x_n$ and $\exists x_n$ are nested but in the last unnested:

$$\forall x_0(P(x_0) \vee \exists x_1 R(x_0, x_1)) \tag{6.1}$$

$$\exists x_0(P(x_0) \wedge \forall x_1 R(x_0, x_1)) \tag{6.2}$$

$$\forall x_0(P(x_0) \vee \exists x_1 Q(x_1)) \tag{6.3}$$

$$(\forall x_0 P(x_0) \vee \exists x_1 Q(x_1)). \tag{6.4}$$

Note that formula (6.3) of quantifier rank 2 is logically equivalent to the formula (6.4) which has quantifier rank 1. So the nesting can sometimes be eliminated. In formulas (6.1) and (6.2) nesting cannot be so eliminated.

Proposition 6.3 *Suppose L is a finite vocabulary without function symbols. For every n and for every set $\{x_1, \ldots, x_n\}$ of variables, there are only finitely many logically non-equivalent first-order L-formulas of quantifier rank $< n$ with the free variables $\{x_1, \ldots, x_n\}$.*

Proof The proof is exactly like that of Proposition 4.15. □

Note that Proposition 6.3 is not true for infinite vocabularies, as there would be infinitely many logically non-equivalent atomic formulas, and also not true for vocabularies with function symbols, as there would be infinitely many logically non-equivalent equations obtained by iterating the function symbols.

6.3 Characterizing Elementary Equivalence

We now show that the concept of a back-and-forth sequence provides an alternative characterization of elementary equivalence

$$\mathcal{A} \equiv \mathcal{B} \quad \text{i.e.} \quad \forall \varphi \in FO(\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi).$$

This is the original motivation for the concepts of a back-and-forth set, back-and-forth sequence, and Ehrenfeucht–Fraïssé Game. To this end, let

$$\mathcal{A} \equiv_n \mathcal{B}$$

mean that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of FO of quantifier rank $\leq n$.

We now prove an important leg of the Strategic Balance of Logic, namely the marriage of truth and separation:

Proposition 6.4 *Suppose L is an arbitrary vocabulary. Suppose $\mathcal{A}$ and $\mathcal{B}$ are L-structures and $n \in \mathbb{N}$. Consider the conditions:*

(i) $\mathcal{A} \equiv_n \mathcal{B}$.
(ii) $\mathcal{A}\restriction_{L'} \simeq_p^n \mathcal{B}\restriction_{L'}$ *for all finite* $L' \subseteq L$.

We have always $(ii) \to (i)$ and if L has no function symbols, then $(ii) \leftrightarrow (i)$.

Proof (ii)→(i). If $\mathcal{A} \not\equiv_n \mathcal{B}$, then there is a sentence φ of quantifier rank $\leq n$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$. Since φ has only finitely many symbols, there is a finite $L' \subseteq L$ such that $\mathcal{A}\restriction_{L'} \not\equiv_n \mathcal{B}\restriction_{L'}$. Suppose $(P_i : i \leq n)$ is a back-and-forth sequence for $\mathcal{A}\restriction_{L'}$ and $\mathcal{B}\restriction_{L'}$. We use induction on $i \leq n$ to prove the following

Claim If $f \in P_i$ and $a_1, \ldots, a_k \in \mathrm{dom}(f)$, then

$$(\mathcal{A}\restriction_{L'}, a_1, \ldots, a_k) \equiv_i (\mathcal{B}\restriction_{L'}, fa_1, \ldots, fa_k).$$

If $i = 0$, the claim follows from $P_0 \subseteq \mathrm{Part}(\mathcal{A}\restriction_{L'}, \mathcal{B}\restriction_{L'})$. Suppose then $f \in P_{i+1}$ and $a_1, \ldots, a_k \in \mathrm{dom}(f)$. Let $\varphi(x_0, x_1, \ldots, x_k)$ be an L'-formula of FO of quantifier rank $\leq i$ such that

$$\mathcal{A}\restriction_{L'} \models \exists x_0 \varphi(x_0, a_1, \ldots, a_k).$$

Let $a \in A$ so that $\mathcal{A}\restriction_{L'} \models \varphi(a, a_1, \ldots, a_k)$ and $g \in P_i$ such that $a \in \mathrm{dom}(g)$ and $f \subseteq g$. By the induction hypothesis, $\mathcal{B}\restriction_{L'} \models \varphi(ga, ga_1, \ldots, ga_k)$. Hence

$$\mathcal{B}\restriction_{L'} \models \exists x_0 \varphi(x_0, fa_1, \ldots, fa_k).$$

The claim is proved. Putting $i = n$ and using the assumption $P_n \neq \emptyset$, gives a contradiction with $\mathcal{A}\restriction_{L'} \not\equiv_n \mathcal{B}\restriction_{L'}$.

(i)$\rightarrow$ (ii). Assume L has no function symbols. Fix $L' \subseteq L$ finite. Let P_i consist of $f : A \rightarrow B$ such that $\mathrm{dom}(f) = \{a_0, \ldots, a_{n-i-1}\}$ and

$$(\mathcal{A}\restriction_{L'}, a_0, \ldots, a_{n-i-1}) \equiv_i (\mathcal{B}\restriction_{L'}, fa_0, \ldots, fa_{n-i-1}).$$

We show that $(P_i : i \leq n)$ is a back-and-forth sequence for $\mathcal{A}\restriction_{L'}$ and $\mathcal{B}\restriction_{L'}$. By (i), $\emptyset \in P_n$ so $P_n \neq \emptyset$. Suppose $f \in P_i, i > 0$, as above, and $a \in A$. By Proposition 6.3 there are only finitely many pairwise non-equivalent L'-formulas of quantifier rank $i - 1$ of the form $\varphi(x, x_0, \ldots, x_{n-i-1})$ in FO. Let them be $\varphi_j(x, x_0, \ldots, x_{n-i-1}), j \in J$. Let

$$J_0 = \{j \in J : \mathcal{A}\restriction_{L'} \models \varphi_j(a, a_0, \ldots, a_{n-i-1})\}.$$

Let

$$\psi(x, x_0, \ldots, x_{n-i-1}) = \bigwedge_{j \in J_0} \varphi_j(x, x_0, \ldots, x_{n-i-1}) \wedge \bigwedge_{j \in J \setminus J_0} \neg\varphi_j(x, x_0, \ldots, x_{n-i-1}).$$

Now $\mathcal{A}\restriction_{L'} \models \exists x \psi(x, a_0, \ldots, a_{n-i-1})$, so as we have assumed $f \in P_i$, we have $\mathcal{B}\restriction_{L'} \models \exists x \psi(x, fa_0, \ldots, fa_{n-i-1})$. Thus there is some $b \in B$ with $\mathcal{B}\restriction_{L'} \models \psi(b, fa_0, \ldots, fa_{n-i-1})$. Now $f \cup \{(a, b)\} \in P_{i-1}$. The other condition (5.15) is proved similarly. □

The above proposition is the standard method for proving models elementary equivalent in FO. For example, Proposition 6.4 and Example 5.26 together give $(Z, <) \equiv (Z + Z, <)$. The exercises give more examples of partially isomorphic pairs – and hence elementary equivalent – structures. The restriction on function symbols can be circumvented by first using quantifiers to eliminate nesting of function symbols and then replacing the unnested equations $f(x_1, \ldots, x_{n-1}) = x_n$ by new predicate symbols $R(x_1, \ldots, x_n)$.

Let $\mathrm{Str}(L)$ denote the class of all L-structures. We can draw the following important conclusion from Proposition 6.4 (see Figure 6.1):

Corollary *Suppose L is a vocabulary without function symbols. Then for all $n \in \mathbb{N}$ the equivalence relation*

$$\mathcal{A} \equiv_n \mathcal{B}$$

divides $\mathrm{Str}(L)$ into finitely many equivalence classes C_i^n, $i = 1, \ldots, m_n$, such that for each C_i^n there is a sentence φ_i^n of FO with the properties:

1. *For all L-structures $\mathcal{A}$: $\mathcal{A} \in C_i^n \iff \mathcal{A} \models \varphi_i^n$.*
2. *If φ is an L-sentence of quantifier rank $\leq n$, then there are $i_1, \ldots, i_k$ such that $\models \varphi \leftrightarrow (\varphi_{i_1}^n \vee \ldots \vee \varphi_{i_k}^n)$.*

Proof Let φ_i^n be the conjunction of all the finitely many L-sentences of quantifier rank $\leq n$ that are true in some (every) model in C_i^n (to make the conjunction finite we do not repeat logically equivalent formulas). For the second claim, let $\varphi_{i_1}^n, \ldots, \varphi_{i_k}^n$ be the finite set of all L-sentences of quantifier rank $\leq n$ that are consistent with φ. If now $\mathcal{A} \models \varphi$, and $\mathcal{A} \in C_i^n$, then $\mathcal{A} \models \varphi_i^n$. On the other hand, if $\mathcal{A} \models \varphi_i^n$ and there is $\mathcal{B} \models \varphi_i^n$ such that $\mathcal{B} \models \varphi$, then $\mathcal{A} \equiv_n \mathcal{B}$, whence $\mathcal{A} \models \varphi$. □

We can actually read from the proof of Proposition 6.4 a more accurate description for the sentences φ_i. This leads to the theory of so-called *Scott formulas* (see Section 7.4).

Theorem 6.5 *Suppose K is a class of L-structures. Then the following are equivalent (see Figure 6.2):*

1. *K is FO-definable, i.e. there is an L-sentence φ of FO such that for all L-structures $\mathcal{M}$ we have $\mathcal{M} \in K \iff \mathcal{M} \models \varphi$.*
2. *There is $n \in \mathbb{N}$ such that K is closed under $\simeq_p^n$.*

As in the case of graphs, Theorem 6.5 can be used to demonstrate that certain properties of models are not definable in FO:

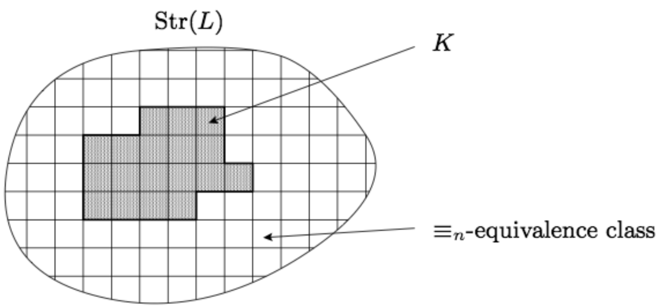


Figure 6.1 First-order definable model class K.

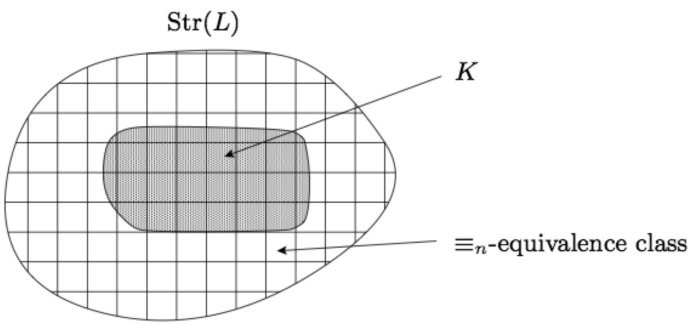


Figure 6.2 Not first-order definable model class K.

Example 6.6 Let $L = \emptyset$. The following properties of L-structures $\mathcal{M}$ are not expressible in FO:

1. M is infinite.
2. M is finite and even.

In both cases it is easy to find, for each $n \in \mathbb{N}$, two models $\mathcal{M}_n$ and $\mathcal{N}_n$ such that $\mathcal{M}_n \simeq_p^n \mathcal{N}_n$, $\mathcal{M}$ has the property, but $\mathcal{N}$ does not.

Example 6.7 Let $L = \{P\}$ be a unary vocabulary. The following properties of L-structures (M, A) are not expressible in FO:

1. $|A| = |M|$.
2. $|A| = |M \setminus A|$.

3. $|A| \leq |M \setminus A|$.

This is demonstrated by the models $(\mathbb{N}, \{1, \ldots, n\})$, $(\mathbb{N}, \mathbb{N} \setminus \{1, \ldots, n\})$, and $(\{1, \ldots, 2n\}, \{1, \ldots, n\})$.

Example 6.8 Let $L = \{<\}$ be a binary vocabulary. The following properties of L-structures $\mathcal{M} = (M, <)$ are not expressible in FO:

1. $\mathcal{M} \cong (\mathbb{Z}, <)$.
2. All closed intervals of $\mathcal{M}$ are finite.
3. Every bounded subset of $\mathcal{M}$ has a supremum.

This is demonstrated in the first two cases by the models $\mathcal{M}_n = (\mathbb{Z}, <)$ and $\mathcal{N}_n = (\mathbb{Z} + \mathbb{Z}, <)$ (see Example 5.26), and in the third case by the partially isomorphic models: $\mathcal{M} = (\mathbb{R}, <)$ and $\mathcal{N} = (\mathbb{R} \setminus \{0\}, <)$.

6.4 The Löwenheim–Skolem Theorem

In this section we show that if a first-order sentence φ is true in a structure $\mathcal{M}$, it is true in a countable substructure of $\mathcal{M}$, and even more, there are countable substructures of $\mathcal{M}$ in a sense "everywhere" satisfying φ. To make this statement precise we introduce a new game from Kueker (1977) called the Cub Game.

Definition 6.9 Suppose A is an arbitrary set. $\mathcal{P}_\omega(A)$ is defined as the set of all countable subsets of A.

The set $\mathcal{P}_\omega(A)$ is an auxiliary concept useful for the general investigation of countable substructures of a model with universe A. One should note that if A is infinite, the set $\mathcal{P}_\omega(A)$ is uncountable.[1] For example, $|\mathcal{P}_\omega(\mathbb{N})| = |\mathbb{R}|$. The set $\mathcal{P}_\omega(A)$ is closed under intersections and countable unions but not necessarily under complements, so it is a (distributive) lattice under the partial order $\subseteq$, but not a Boolean algebra. The sets in $\mathcal{P}_\omega(A)$ cover the set A entirely, but so do many proper subsets of $\mathcal{P}_\omega(A)$ such as the set of all singletons in $\mathcal{P}_\omega(A)$ and the set of all finite sets in $\mathcal{P}_\omega(A)$.

Definition 6.10 Suppose A is an arbitrary set and $\mathcal{C}$ a subset of $\mathcal{P}_\omega(A)$. The *Cub Game of* $\mathcal{C}$ is the game $G_{\text{cub}}(\mathcal{C}) = G_\omega(A, W)$, where W consists of sequences $(a_1, a_2, \ldots)$ with the property that $\{a_1, a_2, \ldots\} \in \mathcal{C}$.

[1] Its cardinality is $|A|^\omega$.

In particular, for every countable $X \subseteq M$ there is a countable submodel $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq N$ and $\mathcal{N} \models T$.

Proof Let $T = \{\varphi_0, \varphi_1, \ldots\}$. By Proposition 6.22 player **II** has a winning strategy in $G_{\text{cub}}(\mathcal{C}_{\varphi_n})$. By Lemma 6.14, player **II** has a winning strategy in $G_{\text{cub}}(\bigcap_{n=0}^{\infty} \mathcal{C}_{\varphi_n})$. If $X \in \bigcap_{n=0}^{\infty} \mathcal{C}_{\varphi_n}$, then $[X]_{\mathcal{M}} \models T$. □

6.5 The Semantic Game

The truth of a first-order sentence in a structure can be defined by means of a simple game called the Semantic Game. We examine this game in detail and give some applications of it.

Definition 6.24 Suppose L is a vocabulary, $\mathcal{M}$ is an L-structure, φ^* is an L-formula, and s^* is an assignment for M. The game $\text{SG}^{\text{sym}}(\mathcal{M}, \varphi^*)$ is defined as follows. In the beginning player **II** holds (φ^*, s^*). The rules of the game are as follows:

1. If φ is atomic, and s satisfies it in $\mathcal{M}$, then the player who holds (φ, s) wins the game, otherwise the other player wins.
2. If $\varphi = \neg\psi$, then the player who holds (φ, s), gives (ψ, s) to the other player.
3. If $\varphi = \psi \wedge \theta$, then the player who holds (φ, s), switches to hold (ψ, s) or (θ, s), and the other player decides which.
4. If $\varphi = \psi \vee \theta$, then the player who holds (φ, s), switches to hold (ψ, s) or (θ, s), and can himself or herself decide which.
5. If $\varphi = \forall x\psi$, then the player who holds (φ, s), switches to hold $(\psi, s[a/x])$ for some a, and the other player decides for which.
6. If $\varphi = \exists x\psi$, then the player who holds (φ, s), switches to hold $(\psi, s[a/x])$ for some a, and can himself or herself decide for which.

As was pointed out in Section 4.2, $\mathcal{M} \models_s \varphi$ if and only if player **II** has a winning strategy in the above game, starting with (φ, s). Why? If $\mathcal{M} \models_s \varphi$, then the winning strategy of player **II** is to play so that if she holds (φ', s'), then $\mathcal{M} \models_{s'} \varphi'$, and if player **I** holds (φ', s'), then $\mathcal{M} \not\models_{s'} \varphi'$.

For practical purposes it is useful to consider a simpler game which presupposes that the formula is in negation normal form. In this game, as in the Ehrenfeucht–Fraïssé Game, player **I** assumes the role of a doubter and player **II** the role of confirmer. This makes the game easier to use than the full game $\text{SG}^{\text{sym}}(\mathcal{M}, \varphi)$.

I	**II**
x_0	
	y_0
x_1	
	y_1
$\vdots$	$\vdots$

Figure 6.11 The game $G_\omega(W)$.

x_n	y_n	Explanation	Rule
$(\varphi, \emptyset)$		**I** enquires about $\varphi \in T$.	
	$(\varphi, \emptyset)$	**II** confirms.	Axiom rule
(φ_i, s)		**I** tests a played $(\varphi_0 \wedge \varphi_1, s)$ by choosing $i \in \{0, 1\}$.	
	(φ_i, s)	**II** confirms.	$\wedge$-rule
$(\varphi_0 \vee \varphi_1, s)$		**I** enquires about a played disjunction.	
	(φ_i, s)	**II** makes a choice of $i \in \{0, 1\}$.	$\vee$-rule
$(\varphi, s[a/x])$		**I** tests a played $(\forall x\varphi, s)$ by choosing $a \in M$.	
	$(\varphi, s[a/x])$	**II** confirms.	$\forall$-rule
$(\exists x\varphi, s)$		**I** enquires about a played existential statement.	
	$(\varphi, s[a/x])$	**II** makes a choice of $a \in M$.	$\exists$-rule

Figure 6.12 The game SG$(\mathcal{M}, T)$.

Definition 6.25 The *Semantic Game* SG$(\mathcal{M}, T)$ of the set T of L-sentences in NNF is the game (see Figure 6.11) $G_\omega(W)$, where W consists of sequences $(x_0, y_0, x_1, y_1, \ldots)$ where player **II** has followed the rules of Figure 6.12 and if player **II** plays the pair (φ, s), where φ is a basic formula, then $\mathcal{M} \models_s \varphi$.

In the game SG$(\mathcal{M}, T)$ player **II** claims that every sentence of T is true in $\mathcal{M}$. Player **I** doubts this and challenges player **II**. He may doubt whether a certain $\varphi \in T$ is true in $\mathcal{M}$, so he plays $x_0 = (\varphi, \emptyset)$. In this round, as in some other rounds too, player **II** just confirms and plays the same pair as player **I**. This may seem odd and unnecessary, but it is for book-keeping purposes only. Player **I** in a sense gathers a finite set of formulas confirmed by player **II** and tries to end up with a basic formula which cannot be true.

Theorem 6.26 *Suppose L is a vocabulary, T is a set of L-sentences, and $\mathcal{M}$ is an L-structure. Then the following are equivalent:*

1. $\mathcal{M} \models T$.
2. *Player* **II** *has a winning strategy in SG$(\mathcal{M}, T)$.*

Proof Suppose $\mathcal{M} \models T$. The winning strategy of player **II** in SG$(\mathcal{M}, T)$ is to maintain the condition $\mathcal{M} \models_{s_i} \psi_i$ for all $y_i = (\psi_i, s_i)$, $i \in \mathbb{N}$, played by her. It is easy to see that this is possible. On the other hand, suppose $\mathcal{M} \not\models T$, say $\mathcal{M} \not\models \varphi$, where $\varphi \in T$. The winning strategy of player **I** in SG$(\mathcal{M}, T)$ is to start with $x_0 = (\varphi, \emptyset)$, and then maintain the condition $\mathcal{M} \not\models_{s_i} \psi_i$ for all $y_i = (\psi_i, s_i)$, $i \in \mathbb{N}$, played by **II**:

1. If $y_i = (\psi_i, s_i)$, where ψ_i is basic, then player **I** has won the game, because $\mathcal{M} \not\models_{s_i} \psi_i$.
2. If $y_i = (\psi_i, s_i)$, where $\psi_i = \theta_0 \wedge \theta_1$, then player **I** can use the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ to find $k < 2$ such that $\mathcal{M} \not\models_{s_i} \theta_k$. Then he plays $x_{i+1} = (\theta_k, s_i)$.
3. If $y_i = (\psi_i, s_i)$, where $\psi_i = \theta_0 \vee \theta_1$, then player **I** knows from the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ that whether **II** plays (θ_k, s_i) for $k = 0$ or $k = 1$, the condition $\mathcal{M} \not\models_{s_i} \theta_k$ still holds. So player **I** can play $x_{i+1} = (\psi_i, s_i)$ and keep his winning criterion in force.
4. If $y_i = (\psi_i, s_i)$, where $\psi_i = \forall x\varphi$, then player **I** can use the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ to find $a \in M$ such that $\mathcal{M} \not\models_{s_i[a/x]} \varphi$. Then he plays $x_{i+1} = (\varphi, s_i[a/x])$.
5. If $y_i = (\psi_i, s_i)$, where $\psi_i = \exists x\varphi$, then player **I** knows from the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ that whatever $(\varphi, s_i[a/x])$ player **II** chooses to play, the condition $\mathcal{M} \not\models_{s_i[a/x]} \varphi$ still holds. So player **I** can play $(\exists x\varphi, s_i)$ and keep his winning criterion in force.

□

Example 6.27 Let $L = \{f\}$ and $\mathcal{M} = (\mathbb{N}, f^{\mathcal{M}})$, where $f(n) = n + 1$. Let

$$\varphi = \forall x \exists y {\approx} fxy.$$

Clearly, $\mathcal{M} \models \varphi$. Thus player **II** has, by Theorem 6.26, a winning strategy in the game SG$(\mathcal{M}, \{\varphi\})$. Figure 6.13 shows how the game might proceed. On

I	II	Rule
$(\forall x\exists y{\approx}fxy, \emptyset)$		
	$(\forall x\exists y{\approx}fxy, \emptyset)$	Axiom rule
$(\exists y{\approx}fxy, \{(x, 25)\})$		
	$(\exists y{\approx}fxy, \{(x, 25)\})$	$\forall$-rule
$(\exists y{\approx}fxy, \{(x, 25)\})$		
	$({\approx}fxy, \{(x, 25), (y, 26)\})$	$\exists$-rule
$\vdots$	$\vdots$	

Figure 6.13 Player **II** has a winning strategy in SG($\mathcal{M}, \{\varphi\}$).

I	II	Rule
$(\forall x\exists y{\approx}fyx, \emptyset)$		
	$(\forall x\exists y{\approx}fyx, \emptyset)$	Axiom rule
$(\exists y{\approx}fyx, \{(x, 0)\})$		
	$(\exists y{\approx}fyx, \{(x, 0)\})$	$\forall$-rule
$(\exists y{\approx}fyx, \{(x, 0)\})$		
	$({\approx}fyx, \{(x, 0), (y, 2)\})$	$\exists$-rule
	(**II** has no good move)	

Figure 6.14 Player **I** wins the game SG($\mathcal{M}, \{\psi\}$).

the other hand, suppose

$$\psi = \forall x\exists y{\approx}fyx.$$

Clearly, $\mathcal{M} \not\models \varphi$. Thus player **I** has, by Theorem 6.26 and Theorem 3.12, a winning strategy in the game SG($\mathcal{M}, \{\varphi\}$). Figure 6.14 shows how the game might proceed.

Example 6.28 Let $\mathcal{M}$ be the graph of Figure 6.15.
and

$$\varphi = \forall x(\exists y\neg xEy \wedge \exists yxEy).$$

Clearly, $\mathcal{M} \models \varphi$. Thus player **II** has, by Theorem 6.26, a winning strategy in the game SG($\mathcal{M}, \{\varphi\}$). Figure 6.16 shows how the game might proceed. On the other hand, suppose

$$\psi = \exists x(\forall y\neg xEy \vee \forall yxEy).$$

Clearly, $\mathcal{M} \not\models \varphi$. Thus player **I** has, by Theorem 6.26 and Theorem 3.12, a winning strategy in the game SG($\mathcal{M}, \{\varphi\}$). Figure 6.17 shows how the game might proceed.

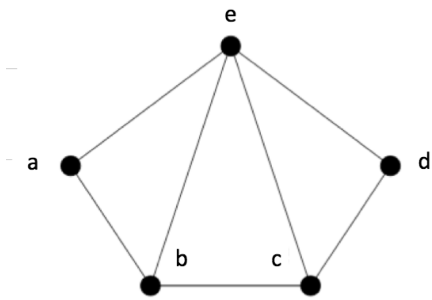


Figure 6.15 The graph $\mathcal{M}$.

I	**II**	Rule
$(\forall x(\exists y \neg xEy \wedge \exists y xEy), \emptyset)$	$(\forall x(\exists y \neg xEy \wedge \exists y xEy), \emptyset)$	Axiom rule
$(\exists y \neg xEy \wedge \exists y xEy, \{(x, d)\})$	$(\exists y \neg xEy \wedge \exists y xEy, \{(x, d)\})$	$\forall$-rule
$(\exists y xEy, \{(x, d)\})$	$(\exists y xEy, \{(x, d)\})$	$\wedge$-rule
$(\exists y xEy, \{(x, d)\})$	$(xEy, \{(x, d), (y, c)\})$	$\exists$-rule
$\vdots$	$\vdots$	

Figure 6.16 Player **II** has a winning strategy in SG($\mathcal{M}$, $\{\varphi\}$).

I	**II**	Rule
$(\exists x(\forall y \neg xEy \vee \forall y xEy), \emptyset)$	$(\exists x(\forall y \neg xEy \vee \forall y xEy), \emptyset)$	Axiom rule
$(\exists x(\forall y \neg xEy \vee \forall y xEy), \emptyset)$	$(\forall y \neg xEy \vee \forall y xEy), \{(x, a)\})$	$\exists$-rule
$(\forall y \neg xEy \vee \forall y xEy, \{(x, a)\})$	$(\forall y \neg xEy, \{(x, a)\})$	$\vee$-rule
$(\neg xEy, \{(x, a), (y, d)\})$	$(\neg xEy, \{(x, a), (y, d)\})$	$\forall$-rule

Figure 6.17 Player **I** wins the game SG($\mathcal{M}$, $\{\psi\}$).

6.6 The Model Existence Game

In this section we learn a new game associated with trying to construct a model for a sentence or a set of sentences. This is of fundamental importance in the sequel.

Let us first recall the game SG($\mathcal{M}, T$): The winning condition for **II** in the game SG($\mathcal{M}, T$) is the only place where the model $\mathcal{M}$ (rather than the set M) appears. If we do not start with a model $\mathcal{M}$ we can replace the winning condition with a slightly weaker one and get a very useful criterion for the existence of *some* $\mathcal{M}$ such that $\mathcal{M} \models T$:

Definition 6.29 The *Model Existence Game* $\mathrm{MEG}(T, L)$ of the set T of L-sentences in NNF is defined as follows. Let C be a countably infinite set of new constant symbols. $\mathrm{MEG}(T, L)$ is the game $G_\omega(W)$ (see Figure 6.11), where W consists of sequences $(x_0, y_0, x_1, y_1, \ldots)$ where player **II** has followed the rules of Figure 6.18 and for no atomic $L \cup C$-sentence φ both φ and $\neg\varphi$ are in $\{y_0, y_1, \ldots\}$.

The idea of the game $\mathrm{MEG}(T, L)$ is that player **I** does not doubt the truth of T (as there is no model around) but rather the mere consistency of T. So he picks those $\varphi \in T$ that he thinks constitute a contradiction and offers them to player **II** for confirmation. Then he runs through the subformulas of these sentences as if there was a model around in which they cannot all be true. He wins if he has made player **II** play contradictory basic sentences. It turns out it did not matter that we had no model around, as two contradictory sentences cannot hold in any model anyway.

Definition 6.30 Let L be a vocabulary with at least one constant symbol. A *Hintikka set (for first-order logic)* is a set H of L-sentences in NNF such that:

1. $\approx tt \in H$ for every constant L-term t.
2. If $\varphi(x)$ is basic, $\varphi(c) \in H$ and $\approx tc \in H$, then $\varphi(t) \in H$.
3. If $\varphi \wedge \psi \in H$, then $\varphi \in H$ and $\psi \in H$.
4. If $\varphi \vee \psi \in H$, then $\varphi \in H$ or $\psi \in H$.
5. If $\forall x\varphi(x) \in H$, then $\varphi(c) \in H$ for all $c \in L$
6. If $\exists x\varphi(x) \in H$, then $\varphi(c) \in H$ for some $c \in L$.
7. For every constant L-term t there is $c \in L$ such that $\approx ct \in H$.
8. There is no atomic sentence φ such that $\varphi \in H$ and $\neg\varphi \in H$.

Lemma 6.31 *Suppose L is a vocabulary and T is a set of L-sentences. If T has a model, then T can be extended to a Hintikka set.*

x_n	y_n	Explanation
φ		**I** enquires about $\varphi \in T$.
	φ	**II** confirms.
$\approx tt$		**I** enquires about an equation.
	$\approx tt$	**II** confirms.
$\varphi(t')$		**I** chooses played $\varphi(t)$ and $\approx tt'$ with φ basic and enquires about substituting t' for t in φ.
	$\varphi(t')$	**II** confirms.
φ_i		**I** tests a played $\varphi_0 \wedge \varphi_1$ by choosing $i \in \{0, 1\}$.
	φ_i	**II** confirms.
$\varphi_0 \vee \varphi_1$		**I** enquires about a played disjunction.
	φ_i	**II** makes a choice of $i \in \{0, 1\}$
$\varphi(c)$		**I** tests a played $\forall x \varphi(x)$ by choosing $c \in C$.
	$\varphi(c)$	**II** confirms.
$\exists x \varphi(x)$		**I** enquires about a played existential statement.
	$\varphi(c)$	**II** makes a choice of $c \in C$
t		**I** enquires about a constant $L \cup C$-term t.
	$\approx ct$	**II** makes a choice of $c \in C$

Figure 6.18 The game $\mathrm{MEG}(T, L)$.

Proof Let us assume $\mathcal{M} \models T$. Let $L' \supseteq L$ such that L' has a constant symbol $c_a \notin L$ for each $a \in M$. Let $\mathcal{M}^*$ be an expansion of $\mathcal{M}$ obtained by interpreting c_a by a for each $a \in M$. Let H be the set of all L'-sentences true in $\mathcal{M}$. It is easy to verify that H is a Hintikka set.

□

Lemma 6.32 *Suppose L is a countable vocabulary and T is a set of L-sentences. If player* **II** *has a winning strategy in* $\mathrm{MEG}(T, L)$*, then the set T can be extended to a Hintikka set in a countable vocabulary extending L by constant symbols.*

Proof Suppose player **II** has a winning strategy in $\mathrm{MEG}(T, L)$. We first run through one carefully planned play of $\mathrm{MEG}(T, L)$. This will give rise to a model $\mathcal{M}$. Then we play again, this time providing a proof that $\mathcal{M} \models T$. To this end, let Trm be the set of all constant $L \cup C$-terms. Let

$$T = \{\varphi_n : n \in \mathbb{N}\},$$
$$C = \{c_n : n \in \mathbb{N}\},$$
$$Trm = \{t_n : n \in \mathbb{N}\}.$$

Let $(x_0, y_0, x_1, y_1, \ldots)$ be a play in which player **II** has used her winning strategy and player **I** has maintained the following conditions:

1. If $n = 3^i$, then $x_n = \varphi_i$.
2. If $n = 2 \cdot 3^i$, then x_n is $\approx c_i c_i$.
3. If $n = 4 \cdot 3^i \cdot 5^j \cdot 7^k \cdot 11^l$, y_i is $\approx t_j t_k$, and y_l is $\varphi(t_j)$, then x_n is $\varphi(t_k)$.
4. If $n = 8 \cdot 3^i \cdot 5^j$, y_i is $\theta_0 \wedge \theta_1$, and $j < 2$, then x_n is θ_j.
5. If $n = 16 \cdot 3^i$, and y_i is $\theta_0 \vee \theta_1$, then x_n is $\theta_0 \vee \theta_1$.
6. If $n = 32 \cdot 3^i \cdot 5^j$, y_i is $\forall x \varphi(x)$, then x_n is $\varphi(c_j)$.
7. If $n = 64 \cdot 3^i$, and y_i is $\exists x \varphi(x)$, then x_n is $\exists x \varphi(x)$.
8. If $n = 128 \cdot 3^i$, then x_n is t_i.

The idea of these conditions is that player **I** challenges player **II** in a maximal way. To guarantee this he makes a plan. The plan is, for example, that on round 3^i he always plays φ_i from the set T. Thus in an infinite game every element of T will be played. Also the plan involves the rule that if player **II** happens to play a conjunction $\theta_0 \wedge \theta_1$ on round i, then player **I** will necessarily play θ_0 on round $8 \cdot 3^i$ and θ_1 on round $8 \cdot 3^i \cdot 5$, etc. It is all just book-keeping – making sure that all possibilities will be scanned. This strategy of **I** is called the *enumeration strategy*. It is now routine to show that $H = \{y_0, y_1, \ldots\}$ is a Hintikka set. □

Lemma 6.33 *Every Hintikka set has a model in which every element is the interpretation of a constant symbol.*

Proof Let $c \sim c'$ if $\approx c'c \in H$. The relation $\sim$ is an equivalence relation on C (see Exercise 6.77). Let us define an $L \cup C$-structure $\mathcal{M}$ as follows.

We let $M = \{[c] : c \in C\}$. For $c \in C$ we let $c^{\mathcal{M}} = [c]$. If $f \in L$ and $\#(f) = n$ we let $f^{\mathcal{M}}([c_{i_1}], \ldots, [c_{i_n}]) = [c]$ for some (any – see Exercise 6.78) $c \in C$ such that $\approx c f c_{i_1} \ldots c_{i_n} \in H$. For any constant term t there is a $c \in C$ such that $\approx ct \in H$. It is easy to see that $t^{\mathcal{M}} = [c]$. For the atomic sentence $\varphi = Rt_1 \ldots t_n$ we let $\mathcal{M} \models \varphi$ if and only if φ is in H. An easy induction on φ shows that if $\varphi(x_1, \ldots, x_n)$ is an L-formula and $\varphi(d_1, \ldots, d_n) \in H$ for some $d_1 \ldots, d_n$, then $\mathcal{M} \models \varphi(d_1, \ldots, d_n)$ (see Exercise 6.79). In particular, $\mathcal{M} \models T$. □

Lemma 6.34 *Suppose L is a countable vocabulary and T is a set of L-sentences. If T can be extended to a Hintikka set in a countable vocabulary extending L, then player* **II** *has a winning strategy in* $\mathrm{MEG}(T, L)$

Proof Suppose L^* is a countable vocabulary extending L such that some Hintikka set H in the vocabulary L^* extends T. Let $C = \{c_n : n \in \mathbb{N}\}$ be a new countable set of constant symbols to be used in $\mathrm{MEG}(T, L)$. Suppose $D = \{t_n : n \in \mathbb{N}\}$ is the set of constant terms of the vocabulary L^*. The winning strategy of player **II** in $\mathrm{MEG}(T, L)$ is to maintain the condition that if y_i is $\varphi(c_1, \ldots, c_n)$, then $\varphi(t_1, \ldots, t_n) \in H$. □

We can now prove the basic element of the Strategic Balance of Logic, namely the following equivalence between the Semantic Game and the Model Existence Game:

Theorem 6.35 (Model Existence Theorem) *Suppose L is a countable vocabulary and T is a set of L-sentences. The following are equivalent:*

1. *There is an L-structure $\mathcal{M}$ such that $\mathcal{M} \models T$.*
2. *Player* **II** *has a winning strategy in* $\mathrm{MEG}(T, L)$.

Proof If there is an L-structure $\mathcal{M}$ such that $\mathcal{M} \models T$, then by Lemma 6.31 there is a Hintikka set $H \supseteq T$. Then by Lemma 6.34 player **II** has a winning strategy in $\mathrm{MEG}(T, L)$. Suppose conversely that player **II** has a winning strategy in $\mathrm{MEG}(T, L)$. By Lemma 6.32 there is a Hintikka set $H \supseteq T$. Finally, this implies by Lemma 6.33 that T has a model. □

Corollary *Suppose L is a countable vocabulary, T a set of L-sentences and φ an L-sentence. Then the following conditions are equivalent:*

1. *$T \models \varphi$.*
2. *Player* **I** *has a winning strategy in* $\mathrm{MEG}(T \cup \{\neg\varphi\}, L)$.

Proof By Theorem 3.12 the game $\mathrm{MEG}(T \cup \{\neg\varphi\}, L)$ is determined. So by Theorem 6.35, condition 2 is equivalent to $T \cup \{\neg\varphi\}$ not having a model, which is exactly what condition 1 says. □

Condition 1 of the above Corollary is equivalent to φ having a *formal proof* from T. (See Enderton (2001), or any standard textbook in logic for a definition of formal proof.) We can think of a winning strategy of player **I** in $\mathrm{MEG}(T \cup \{\neg\varphi\}, L)$ as a *semantic proof*. In the literature this concept occurs under the names *semantic tree* or *Beth tableaux*.

6.7 Applications

The Model Existence Theorem is extremely useful in logic. Our first application – The Compactness Theorem – is a kind of model existence theorem itself and very useful throughout model theory.

Theorem 6.36 (Compactness Theorem) *Suppose L is a countable vocabulary and T is a set of L-sentences such that every finite subset of T has a model. Then T has a model.*

Proof Let C be a countably infinite set of new constant symbols as needed in $\mathrm{MEG}(T, L)$. The winning strategy of player **II** in $\mathrm{MEG}(T, L)$ is the following. Suppose

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

has been played up to now, and then player **I** plays x_n. Player **II** has made sure that $T \cup \{y_0, \ldots, y_{n-1}\}$ is *finitely consistent*, i.e. each of its finite subsets has a model. Now she makes such a move y_n that $T \cup \{y_0, \ldots, y_n\}$ is still finitely consistent. Suppose this is the case and player **I** asks a confirmation for φ, where $\varphi \in T$. Now $T \cup \{y_0, \ldots, y_{n-1}, \varphi\}$ is finitely consistent as it is the same set as $T \cup \{y_0, \ldots, y_{n-1}\}$. Suppose then player **I** asks a confirmation for θ_0, where $\theta_0 \wedge \theta_1 = y_i$ for some $i < n$. If $T_0 \cup \{y_0, \ldots, y_{n-1}, \theta_0\}$ has no model, where T_0 is a finite subset of T, then surely $T_0 \cup \{y_0, \ldots, y_{n-1}\}$ has no models either, a contradiction. Suppose then player **I** asks for a decision about $\theta_0 \vee \theta_1$, where $\theta_0 \vee \theta_1 = y_i$ for some $i < n$. If $T_0 \cup \{y_0, \ldots, y_{n-1}, \theta_0\}$ has no models, where T_0 is a finite subset of T, and also $T_1 \cup \{y_0, \ldots, y_{n-1}, \theta_1\}$ has no models, where T_1 is another finite subset of T, then $T_0 \cup T_1 \cup \{y_0, \ldots, y_{n-1}\}$ has no models, a contradiction. Suppose then player **I** asks for a confirmation for $\varphi(c)$, where $\forall x \varphi(x) = y_i$ for some $i < n$ and $c \in C$. If $T_0 \cup \{y_0, \ldots, y_{n-1}, \varphi(c)\}$ has no models, where T_0 is a finite subset of T, then $T_0 \cup \{y_0, \ldots, y_{n-1}\}$ has no models either, a contradiction. Suppose then player **I** asks a decision about $\exists x \varphi(x)$, where $\exists x \varphi(x) = y_i$ for some $i < n$. Let $c \in C$ so that c does not occur in $\{y_0, \ldots, y_{n-1}\}$. We claim that $T \cup \{y_0, \ldots, y_{n-1}, \varphi(c)\}$ is finitely consistent. Suppose the contrary. Then there is a finite conjunction ψ of sentences in T such that

$$\{y_0, \ldots, y_{n-1}, \psi\} \models \neg\varphi(c).$$

Hence

$$\{y_0, \ldots, y_{n-1}, \psi\} \models \forall x \neg\varphi(x).$$

But this contradicts the fact that $\{y_0, \ldots, y_{n-1}, \psi\}$ has a model in which $\exists x\varphi(x)$ is true. Finally, if t is a constant term, it follows as above that there is a constant $c \in C$ such that $T \cup \{y_0, \ldots, y_{n-1}, \approx ct\}$ is finitely consistent. □

It is a consequence of the Compactness Theorem that a theory in a countable vocabulary is consistent in the sense that every finite subset has a model if and only if it is consistent in the sense that T itself has a model. Therefore the word "consistent" is used in both meanings.

As an application of the Compactness Theorem consider the vocabulary $L = \{+, \cdot, 0, 1\}$ of number theory. An example of an L-structure is the so-called *standard model of number theory* $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1)$. L-structures may be elementary equivalent to $\mathcal{N}$ and still be *non-standard* in the sense that they are not isomorphic to $\mathcal{N}$. Let c be a new constant symbol. It is easy to see that the theory

$$\{\varphi : \mathcal{N} \models \varphi\} \cup \{1 < c, +11 < c, ++111 < c, \ldots\}$$

is finitely consistent. By the Compactness Theorem it has a model $\mathcal{M}$. Clearly $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M} \not\cong \mathcal{N}$.

Example 6.37 Suppose T is a theory in a countable vocabulary L, and T has for each $n > 0$ a model $\mathcal{M}_n$ such that $(M_n, E^{\mathcal{M}_n})$ is a graph with a cycle of length $\geq n$. We show that T has a model $\mathcal{N}$ such that $(N, E^{\mathcal{N}})$ is a graph with an infinite cycle (i.e. an infinite connected subgraph in which every node has degree 2). To this end, let $c_z, z \in \mathbb{Z}$, be new constant symbols. Let T' be the theory

$$T \cup \{c_z E c_{z+1} : z \in \mathbb{Z}\}.$$

Any finite subset of T' mentions only finitely constants c_z, so it can be satisfied in the model $\mathcal{M}_n$ for a sufficiently large n. By the Compactness Theorem T' has a model $\mathcal{M}$. Now $\mathcal{M} \upharpoonright L \models T$ and the elements $c_z^{\mathcal{M}}$, $z \in \mathbb{Z}$, constitute an infinite cycle in $\mathcal{M}$.

As another application of the Model Existence Game we prove the so-called

Thus

$$(\mathbb{N}, +, \cdot, 0, 1, A) \equiv \mathcal{M}' \equiv \mathcal{M}.$$

□

In general, the significance of the Omitting Types Theorem is the fact that it can be used – as above – to get "standard" models.

6.8 Interpolation

The Craig Interpolation Theorem says the following: Suppose $\models \varphi \to \psi$, where φ is an L_1-sentence and ψ is an L_2-sentence. Then there is an $L_1 \cap L_2$-sentence θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$. Here is an example:

Example 6.39 $L_1 = \{P, Q, R\}, L_2 = \{P, Q, S\}$. Let

$$\varphi = \forall x(Px \to Rx) \land \forall x(Rx \to Qx)$$

and

$$\psi = \forall x(Sx \to Px) \to \forall x(Sx \to Qx).$$

Now

$$\models \varphi \to \psi,$$

and indeed, if

$$\theta = \forall x(Px \to Qx),$$

then θ is an $L_1 \cap L_2$-sentence such that

$$\models \varphi \to \theta \text{ and } \models \theta \to \psi.$$

The Craig Interpolation Theorem is a consequence of the following remarkable *subformula property* of the Model Existence Game $\mathrm{MEG}(T, L)$: Player **II** never has to play anything but subformulas of sentences of T up to a substitution of terms for free variables.

Theorem 6.40 (Craig Interpolation Theorem) *Suppose $\models \varphi \to \psi$, where φ is an L_1-sentence and ψ is an L_2-sentence. Then there is an $L_1 \cap L_2$-sentence θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$.*

Proof We assume, for simplicity, that L_1 and L_2 are relational. This restriction can be avoided (see Exercise 6.97). Let us assume that the claim of the theorem is false and derive a contradiction. Since $\models \varphi \to \psi$, player **I** has a winning strategy in $\mathrm{MEG}(\{\varphi, \neg\psi\}, L_1 \cup L_2)$. Therefore to reach a contradiction it suffices to construct a winning strategy for player **II** in $\mathrm{MEG}(\{\varphi, \neg\psi\}, L_1 \cup L_2)$. If φ alone is inconsistent, we can take any inconsistent L-sentence as θ. Likewise if $\neg\psi$ alone is inconsistent, we can take any valid L-sentence as θ. Let $L = L_1 \cap L_2$. Let us consider the following strategy of player **II**. Suppose $C = \{c_n : n \in \mathbb{N}\}$ is a set of new constant symbols. We denote $L \cup C$-sentences by $\theta(c_0, \ldots, c_{m-1})$ where $\theta(z_0, \ldots, z_{m-1})$ is assumed to be an L-formula. Suppose player **II** has played $Y = \{y_0, \ldots, y_{n-1}\}$ so far. While she plays, she maintains two subsets S_1^n and S_2^n of Y such that $S_1^n \cup S_2^n = Y$. The set S_1^n consists of all $L_1 \cup C$-sentences in Y, and S_2^n consists of all $L_2 \cup C$-sentences in Y. Let us say that an $L \cup C$-sentence θ *separates* S_1^n and S_2^n if $S_1^n \models \theta$ and $S_2^n \models \neg\theta$. Player **II** plays so that the following condition holds at all times:

($\star$) There is no $L \cup C$-sentence θ that separates S_1^n and S_2^n.

Let us check that she can maintain this strategy: (There is no harm in assuming that player **I** plays φ and $\neg\psi$ first.)

Case 1. Player **I** plays φ. We let $S_1^0 = \{\varphi\}$ and $S_2^0 = \emptyset$. Condition ($\star$) holds, as S_1^n is consistent.

Case 2. Player **I** plays $\neg\psi$ having already played φ. We let $S_1^1 = \{\varphi\}$ and $S_2^1 = \{\neg\psi\}$. Suppose $\theta(c_0, \ldots, c_{m-1})$ separates S_1^1 and S_2^1. Then $\models \varphi \rightarrow \forall z_0 \ldots \forall z_{m-1}\theta(z_0, \ldots, z_{m-1})$ and $\models \forall z_0 \ldots \forall z_{m-1}\theta(z_0, \ldots, z_{m-1}) \rightarrow \psi$ contrary to assumption.

Case 3. Player **I** plays $\approx cc$, where, for example, $c \in L_1 \cup C$. We let $S_1^{n+1} = S_0^n \cup \{\approx cc\}$ and $S_2^{n+1} = S_1^n \cup \{\approx cc\}$. Suppose $\theta(c_0, \ldots, c_{m-1})$ separates S_1^{n+1} and S_2^{n+1}. Then clearly also $\theta(c_0, \ldots, c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 4. Player **I** plays $\varphi_0(c_1)$, where, for example, $\varphi_0(c_0), \approx c_0c_1 \in S_1^n$. We let $S_1^{n+1} = S_1^n \cup \{\varphi_0(c_1)\}$ and $S_2^{n+1} = S_2^n$. Suppose $\theta(c_0, \ldots, c_m)$ separates S_1^{n+1} and S_2^{n+1}. Then as $S_1^n \models \varphi_0(c_1)$ clearly $\theta(c_0, \ldots, c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 5. Player **I** plays φ_i, where, for example, $\varphi_1 \wedge \varphi_1 \in S_1^n$. We let $S_1^{n+1} = S_1^n \cup \{\varphi_i\}$ and $S_2^{n+1} = S_2^n$. Suppose $\theta(c_0, \ldots, c_{m-1})$ separates S_1^{n+1} and S_2^{n+1}. Then, as $S_1^n \models \varphi_i$, clearly $\theta(c_0, \ldots, c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 6. Player **I** plays $\varphi_0 \vee \varphi_1$, where, for example, $\varphi_0 \vee \varphi_1 \in S_1^n$. We claim that for one of $i \in \{0, 1\}$ the sets $S_1^n \cup \{\varphi_i\}$ and S_2^n satisfy ($\star$). Otherwise there is for both $i \in \{0, 1\}$ some $\theta_i(c_0, \ldots, c_{m-1})$ that separates $S_1^n \cup \{\varphi_i\}$

and S_2^n. Let

$$\theta(c_0, \ldots, c_{m-1}) = \theta_0(c_0, \ldots, c_{m-1}) \vee \theta_1(c_0, \ldots, c_{m-1}).$$

Then, as $S_1^n \models \varphi_0 \vee \varphi_1$, clearly $\theta(c_0, \ldots, c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 7. Player **I** plays $\varphi(c_0)$, where, for example, $\forall x\varphi(x) \in S_1^n$. We claim that the sets $S_1^n \cup \{\varphi(c_0)\}$ and S_2^n satisfy ($\star$). Otherwise there is $\theta(c_0, \ldots, c_{m-1})$ that separates $S_1^n \cup \{\varphi(c_0)\}$ and S_2^n. Let

$$\theta'(c_1, \ldots, c_{m-1}) = \forall x\theta(x, c_1, \ldots, c_{m-1}).$$

Then, as $S_1^n \models \forall x\varphi(x)$, we have $S_1^n \models \varphi(c_0)$, and hence $\theta'(c_0, c_1, \ldots, c_{m-1})$ separates S_1^n and S_2^n, a contradiction.

Case 8. Player **I** plays $\exists x\varphi(x)$, where, for example, $\exists x\varphi(x) \in S_1^n$. Let $c \in C$ be such that c does not occur in Y yet. We claim that the sets $S_1^n \cup \{\varphi(c)\}$ and S_2^n satisfy ($\star$). Otherwise there is some $\theta(c, c_0, \ldots, c_{m-1})$ that separates $S_1^n \cup \{\varphi(c)\}$ and S_2^n. Let

$$\theta'(c_1, \ldots, c_{m-1}) = \exists x\theta(x, c_0, \ldots, c_{m-1}).$$

Then, as $S_1^n \models \exists x\varphi(x)$ and $S_1^n \models \varphi(c) \to \theta(c, c_0, \ldots, c_{m-1})$ we clearly have that $\theta'(c_1, \ldots, c_{m-1})$ separates S_1^n and S_2^n, a contradiction. □

Example 6.41 The Craig Interpolation Theorem is false in finite models. To see this, let $L_1 = \{R\}$ and $L_2 = \{P\}$ where R and P are distinct binary predicates. Let φ say that R is an equivalence relation with all classes of size 2 and let ψ say P is not an equivalence relation with all classes of size 2 except one of size 1. Then $\mathcal{M} \models \varphi \to \psi$ holds for finite $\mathcal{M}$. If there were a sentence θ of the empty vocabulary such that $\mathcal{M} \models \varphi \to \theta$ and $\mathcal{M} \models \theta \to \psi$ for all finite $\mathcal{M}$, then θ would characterize even cardinality in finite models. It is easy to see with Ehrenfeucht–Fraïssé Games that this is impossible.

Theorem 6.42 (Beth Definability Theorem) *Suppose L is a vocabulary and P is a predicate symbol not in L. Let φ be an $L \cup \{P\}$-sentence. Then the following are equivalent:*

1. *If $(\mathcal{M}, A) \models \varphi$ and $(\mathcal{M}, B) \models \varphi$, where $\mathcal{M}$ is an L-structure, then $A = B$.*
2. *There is an L-formula θ such that*

$$\varphi \models \forall x_0 \ldots x_{n-1}(\theta(x_0, \ldots, x_{n-1}) \leftrightarrow P(x_0, \ldots, x_{n-1})).$$

If condition 1 holds we say that φ defines P *implicitly*. If condition 2 holds, we say that θ defines P *explicitly* relative to φ.

Proof Let φ' be obtained from φ by replacing everywhere P by P' (another new predicate symbol). Then condition 1 implies

$$\models (\varphi \wedge Pc_0 \ldots c_{n-1}) \to (\varphi' \to P'c_0 \ldots c_{n-1}).$$

By the Craig Interpolation Theorem there is an L-formula $\theta(x_0, \ldots, x_{n-1})$ such that

$$\models (\varphi \wedge Pc_0 \ldots c_{n-1}) \to \theta(c_0, \ldots, c_{n-1})$$

and

$$\models \theta(c_0, \ldots, c_{n-1}) \to (\varphi' \to P'c_0 \ldots c_{n-1}).$$

It follows easily that θ is the formula we are looking for. □

Example 6.43 The Beth Definability Theorem is false in finite models. Let φ be the conjunction of

1. "$<$ is a linear order".
2. $\exists x(Px \wedge \forall y(\approx xy \vee x < y))$.
3. $\forall x \forall y($"y immediate successor of x" $\to (Px \leftrightarrow \neg Py))$.

Every finite linear order has a unique P with φ, but there is no $\{<\}$-formula $\theta(x)$ which defines P in models of φ. For then the sentence

$$\exists x(\theta(x) \wedge \forall y(\approx xy \vee y < x))$$

would characterize ordered sets of odd length among finite ordered sets, and it is easy to see with Ehrenfeucht–Fraïssé Games that no such sentence can exist. There are infinite linear orders (e.g. $(\mathbb{N} + \mathbb{Z}, <)$) where several different P satisfy φ.

Recall that the *reduct* of an L-structure $\mathcal{M}$ to a smaller vocabulary K is the structure $\mathcal{N} = \mathcal{M} \upharpoonright K$ which has M as its universe and the same interpretations of all symbols of K as $\mathcal{M}$. In such a case we call $\mathcal{M}$ an *expansion* of $\mathcal{N}$ from vocabulary K to vocabulary L. Another useful operation on structures is the following. The *relativization* of an L-structure $\mathcal{M}$ to a set N is the structure $\mathcal{N} = \mathcal{M}^{(N)}$ which has N as its universe, $R^{\mathcal{M}} \cap N^{\#(R)}$ as the interpretation of any predicate symbol $R \in L$, $f^{\mathcal{M}} \upharpoonright N^{\#(f)}$ as the interpretation of any function symbol $f \in L$, and $c^{\mathcal{M}}$ as the interpretation of any constant symbol $c \in L$. Relativization is only possible when the result actually *is* an L-structure. There is a corresponding operation on formulas: The *relativization* of an L-formula φ to a predicate $P \in L$ is defined by replacing every quantifier $\forall y \ldots$ in φ by $\forall y(Py \to \ldots)$ and every quantifier $\exists y \ldots$ in φ by $\exists y(Py \wedge \ldots)$. We denote the relativization by $\psi^{(P)}$.

Lemma 6.44 *Suppose L is a relational vocabulary and $P \in L$ is a unary predicate symbol. The following are equivalent for all L-formulas φ and all L-structures $\mathcal{M}$ such that $P^{\mathcal{M}} \neq \emptyset$:*

1. $\mathcal{M} \models \varphi^{(P)}$.
2. $\mathcal{M}^{(P^{\mathcal{M}})} \models \varphi$.

Proof Exercise 6.101. □

Definition 6.45 Suppose L is a vocabulary. A class K of L-structures is an *EC-class* if there is an L-sentence φ such that

$$K = \{\mathcal{M} \in \mathrm{Str}(L) : \mathcal{M} \models \varphi\}$$

and a *PC-class* if there is an L'-sentence φ for some $L' \supseteq L$ such that

$$K = \{\mathcal{M} \restriction L : \mathcal{M} \in \mathrm{Str}(L') \text{ and } \mathcal{M} \models \varphi\}.$$

Example 6.46 Let $L = \emptyset$. The class of infinite L-structures is a PC-class which is not an EC class. (Exercise 6.102.)

Example 6.47 Let $L = \emptyset$. The class of finite L-structures is not a PC-class. (Exercise 6.103.)

Example 6.48 Let $L = \{<\}$. The class of non-well-ordered L-structures is a PC-class which is not an EC-class. (Exercise 6.104.)

Suppose $\models \varphi \to \psi$, where φ is an L_1-sentence and ψ is an L_2-sentence. Let

$$K_1 = \{\mathcal{M} \restriction (L_1 \cap L_2) : \mathcal{M} \models \varphi\}$$

and

$$K_2 = \{\mathcal{M} \restriction (L_1 \cap L_2) : \mathcal{M} \models \neg\psi\}.$$

Now K_1 and K_2 are disjoint PC-classes. If there is an $L_1 \cap L_2$-sentence θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$, then the EC-class

$$K = \{\mathcal{M} : \mathcal{M} \models \theta\}$$

separates K_1 and K_2 in the sense that $K_1 \subseteq K$ and $K_2 \cap K = \emptyset$. On the other hand, if an EC-class K separates in this sense K_1 and K_2, then there is an $L_1 \cap L_2$-sentence θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$. Thus the Craig Interpolation Theorem can be stated as: disjoint PC-classes can always be separated by an EC-class.

Theorem 6.49 (Separation Theorem) *Suppose K_1 and K_2 are disjoint PC-classes of models. Then there is an EC-class K that separates K_1 and K_2, i.e. $K_1 \subseteq K$ and $K_2 \cap K = \emptyset$.*

Proof The claim has already been proved in Theorem 6.40, but we give here a different – model-theoretic – proof. This proof is of independent interest, being as it is, in effect, the proof of the so-called *Lindström's Theorem* (Lindström (1973)), which gives a model theoretic characterization of first order logic.

Case 1: There is an $n \in \mathbb{N}$ such that some union K of $\simeq_p^n$-equivalence classes of models separates K_1 and K_2. By Theorem 6.5 the model class K is an EC-class, so the claim is proved.

Case 2: There are, for any $n \in \mathbb{N}$, $L_1 \cap L_2$-models $\mathcal{M}_n$ and $\mathcal{N}_n$ such that $\mathcal{M}_n \in K_1$, $\mathcal{N}_n \in K_2$, and there is a back-and-forth sequence $(I_i : i \leq n)$ for $\mathcal{M}_n$ and $\mathcal{N}_n$. Suppose K_1 is the class of reducts of models of φ, and K_2 respectively the class of reducts of models of ψ. Let T be the following set of sentences:

1. $\varphi^{(P_1)}$.
2. $\psi^{(P_2)}$.
3. $(R, <)$ is a non-empty linear order in which every element with a predecessor has an immediate predecessor.
4. $\forall z(Rz \to Q_0 z)$.
5. $\forall z \forall u_1 \ldots \forall u_m \forall v_1 \ldots \forall v_m((Rz \wedge Q_n z u_1 \ldots u_m v_1 \ldots v_m) \to (\theta(u_1, \ldots, u_m) \leftrightarrow \theta(v_1, \ldots, v_m)))$ for all atomic $L_1 \cap L_2$-formulas θ.
6. $\forall z \forall u_1 \ldots \forall u_n \forall v_1 \ldots \forall v_m((Rz \wedge Q_n z u_1 \ldots u_m v_1 \ldots v_m) \to \forall z' \forall x((Rz' \wedge z' < z \wedge \forall w(w < z \to (w < z' \vee w = z')) \wedge P_1 x) \to \exists y(P_2 y \wedge Q_{n+1} z' u_1 \ldots u_m x v_1 \ldots v_m y)))$.
7. $\forall z \forall u_1 \ldots \forall u_m \forall v_1 \ldots \forall v_m((Rz \wedge Q_n z u_1 \ldots u_m v_1 \ldots v_m) \to \forall z' \forall y((Rz' \wedge z' < z \wedge \forall w(w < z \to (w < z' \vee w = z')) \wedge P_2 y) \to \exists x(P_1 x \wedge Q_{n+1} z' u_1 \ldots u_m x v_1 \ldots v_m y)))$.

For all $n \in \mathbb{N}$ there is a model $\mathcal{A}_n$ of T with $(R, <)$ of length n. The model $\mathcal{A}_n$ is obtained as follows. The universe A_n is the (disjoint) union of M_n, N_n, and $\{1, \ldots, n\}$. The L_1-structure $(\mathcal{A}_n \restriction L_1)^{P_1^{\mathcal{A}_n}}$ is chosen to be a copy of the model $\mathcal{M}_n$ of φ. The L_2-structure $(\mathcal{A}_n \restriction L_2)^{P_2^{\mathcal{A}_n}}$ is chosen to be a copy of the model $\mathcal{N}_n$ of ψ. The $2i+1$-ary predicate Q_i is interpreted in $\mathcal{A}_n$ as the set

$$\{(n-i, u_1, \ldots, u_i, v_1, \ldots, v_i) : \{(u_1, v_1), \ldots, (u_i, v_i)\} \in I_{n-i}\}.$$

By the Compactness Theorem, there is a countable model $\mathcal{M}$ of T with $(R, <)$ non-well-founded (see Exercise 6.107). That is, there are $a_n, n \in \mathbb{N}$, in M such that a_{n+1} is an immediate predecessor of a_n in $\mathcal{M}$ for all $n \in \mathbb{N}$. Let $\mathcal{M}_1$ be the $L_1 \cap L_2$-structure $(\mathcal{M} \restriction (L_1 \cap L_2))^{(P_1^{\mathcal{M}})}$. Let $\mathcal{M}_2$ be the $L_1 \cap L_2$-structure $(\mathcal{M} \restriction (L_1 \cap L_2))^{(P_2^{\mathcal{M}})}$. Now $\mathcal{M}_1 \simeq_p \mathcal{M}_2$, for we have the back-and-forth set:

$$P = \{\{(u_1, v_1), \ldots, (u_n, v_n)\} : \mathcal{M} \models Q_n a_n u_1 \ldots u_n v_1 \ldots v_n, n \in \mathbb{N}\}.$$

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable, they are isomorphic. But $\mathcal{M}_1 \in K_1$ and $\mathcal{M}_2 \in K_2$, a contradiction. □

6.9 Uncountable Vocabularies

So far we have concentrated on methods based on the assumption that vocabularies are countable. Several key methods work also for uncountable vocabularies. A typical application of uncountable vocabularies is the task of finding an elementary extension of an uncountable structure. In this case a new constant symbol is added to the vocabulary for each element of the model, and the vocabulary may become uncountable.

Strictly speaking, handling uncountable vocabularies does not require dealing with ordinals, but since we use the Axiom of Choice anyway, it is more natural to assume our vocabularies are well-ordered as in

$$L = \{R_\alpha : \alpha < \beta\} \cup \{f_\alpha : \alpha < \gamma\} \cup \{c_\alpha : \alpha < \delta\}.$$

We then allow also variable symbols $x_\alpha, \alpha < \epsilon$.

An important method throughout logic is the method of Skolem functions.

Definition 6.50 Suppose L is a vocabulary, $\mathcal{M}$ is an L-structure, and we have an L-formula $\varphi(x_0, \ldots, x_n)$ of first-order logic. A *Skolem function* for $\varphi(x_0, \ldots, x_n)$ in $\mathcal{M}$ is any function $f_\varphi : M^n \to M$ such that for all elements $a_0, \ldots, a_{n-1}$ of M:

$$M \models \exists x_n \varphi(a_0, \ldots, a_{n-1}, x_n) \to \varphi(a_0, \ldots, a_{n-1}, f_\varphi(a_0, \ldots, a_{n-1})).$$

The following simple but fundamental fact is very helpful in the applications of Skolem functions:

Proposition 6.51 (Tarski–Vaught criterion) *Suppose L is a vocabulary, $\mathcal{M}$ an L-structure, and $\mathcal{N} \subseteq \mathcal{M}$ such that for all L-formulas $\varphi(x_0, \ldots, x_n)$ the following holds:*

If $a_0, \ldots, a_{n-1} \in N$ and $\mathcal{M} \models \varphi(a_0, \ldots, a_{n-1}, a_n)$ for some $a_n \in M$, then $\mathcal{M} \models \varphi(a_0, \ldots, a_{n-1}, a'_n)$ for some $a'_n \in N$. (6.7)

Then $\mathcal{N} \prec \mathcal{M}$.

Proof Exercise 6.110. □

Proposition 6.52 *Suppose L is a vocabulary, $\mathcal{M}$ an L-structure, and $\mathcal{F}$ a family of functions such that every L-formula has a Skolem function $\in \mathcal{F}$ in* formula of the vocabulary L_m. By definition,

$$J = \{i \in I \ \ \mathcal{M}_i \models \varphi(a_0(i), \ldots, a_{n-1}(i))\} \in D.$$

Suppose $i \in J \cap E_m$. Again, $n_i \geq m$. Now

$$\mathcal{M}_i \models \varphi(a_0(i), \ldots, a_{n-1}(i))$$

and (6.21) is a position in the game $EF_{n_i}(\mathcal{M}_i \restriction L_{n_i}, \mathcal{N}_i \restriction L_{n_i})$ while **II** uses τ_i. Since τ_i is a winning strategy of **II** and $\varphi(x_0, \ldots, x_{n-1})$ is a formula of the vocabulary L_{n_i},

$$\mathcal{N}_i \models \varphi(b_0(i), \ldots, b_{n-1}(i)).$$

Thus

$$\{i \in I \ \ \mathcal{N}_i \models \varphi(b_0(i), \ldots, b_{n-1}(i))\} \supseteq J \cap E_m \in D,$$

whence $\mathcal{N} \models \varphi(b_0, \ldots, b_{n-1})$. □

Theorem 6.68 is by no means the best in this direction (see Benda (1969)). A particularly beautiful stronger result is the following result of Shelah (1971): $\mathcal{M} \equiv \mathcal{N}$ if and only if there are I and an ultrafilter D on I such that $\prod_i \mathcal{M}/D \cong \prod_i \mathcal{N}/D$.

6.11 Historical Remarks and References

Basic texts in model theory are Chang and Keisler (1990) and Hodges (1993). For the history of model theory, see Vaught (1974). Characterization of elementary equivalence in terms of back-and-forth sequences (Theorem 6.5 and its Corollary) is due to Fraïssé (1955).

The concepts and results of Section 6.4 are due to Kueker (1972, 1977). Theorem 6.23 goes back to Löwenheim (1915) and Skolem (1923, 1970).

The idea of interpreting the quantifiers in terms of moves in a game, as in the Semantic Game, is due to Henkin (1961). Hintikka (1968) extended this from quantifiers to propositional connectives and emphasized its role in semantics in general. The roots of interpreting logic as a game go back, arguably, to Wittgenstein's language games. Lorenzen (1961) used a similar game in proof theory. For the close general connection between inductive definitions and games see Aczel (1977).

Our Model Existence Game is a game-theoretic rendering of the method of semantic tableaux of Beth (1955a,b), model sets of Hintikka (1955), and consistency properties of Smullyan (1963, 1968). Its roots are in the proof-theoretic method of natural deduction of Gentzen (1934, 1969). A good source for more advanced applications of the Model Existence Game is Hodges (1985). Theorem 6.42 is due to Beth (1953) and the stronger but related Theorem 6.40 to Craig (1957a). For the background of Theorem 6.40 see Craig (2008), and for its early applications Craig (1957b). The failure of Graig Interpolation in finite models was observed in Hájek (1976), see also Gurevich (1984), which has this and Example 6.43. The proof of Theorem 6.49 is modeled according to the proof in Barwise and Feferman (1985) of the main result of Lindström (1973), the so-called Lindström's Theorem, which characterizes first-order logic as a maximal logic which satisfies the Compactness Theorem and the Löwenheim–Skolem Theorem in the form: every sentence of the logic which has an infinite model has a countable model. The connection to Theorem 6.49 is the following: Suppose L^* were such a logic and $\varphi \in L^*$. We could treat the class of models of φ and the class of models of $\neg\varphi$ as we treat the disjoint PC-classes K_1 and K_2 in Theorem 6.49. The proof then shows that a first-order sentence θ can separate the class of models of φ and the class of models of $\neg\varphi$. This would clearly mean that φ would be logically equivalent to θ, hence first-order definable. Theorem 6.62 is from Keisler and Morley (1968).

Ultraproducts were introduced by Łoś (1955). For a survey of the use of them in model theory see Bell and Slomson (1969). Theorem 6.68 is from Benda (1969).

Exercise 6.9 is from Brown and Hoshino (2007), where more information about Ehrenfeucht-Fraïssé games for paths and cycles can be found. See also Bissell-Siders (2007). Exercise 6.114 is from Morley (1968).

Exercises

6.1 A finite connected graph is a *cycle* if every vertex has degree 2. Write a sentence of quantifier rank 2 which holds in a cycle if and only if the cycle has length 3. Show that no such sentence of quantifier rank 1 exists.

6.2 Write a sentence of quantifier rank 3 which holds in a cycle if and only if the cycle has length 4. Show that no such sentence of quantifier rank 2 exists. Do the same for the cycle of length 5.

6.3 Do the previous Exercise for the cycle of length 6.

6.4 Write a sentence of quantifier rank 4 which holds in a graph if and only if the cycle has length 7. Show that no such sentence of quantifier rank 3 exists. Do the same for the cycle of length 8.

6.5 Write a sentence of quantifier rank 4 which holds in a graph if and only if the cycle has length 9. Show that no such sentence of quantifier rank 3 exists.

6.6 Construct a sentence of quantifier rank 2 which is true in an ordered set $\mathcal{M}$ if and only if $\mathcal{M}$ has length 2.

6.7 Construct a sentence φ_n of quantifier rank 3 which is true in an ordered set $\mathcal{M}$ if and only if $\mathcal{M}$ has length n, where n is $3, 4, 5$, or 6.

6.8 Show that there is a sentence of quantifier rank 4 which is true in a graph if and only if the graph is a cycle, but no such sentence of quantifier rank 3 exists.

6.9 Show that if $n \geq 3$ and $\mathcal{M}$ and $\mathcal{N}$ are cycles of length $\geq 2^{n-1} + 3$, then $\mathcal{M} \simeq_p^n \mathcal{N}$.

6.10 Suppose $L = \emptyset$ and $n \in \mathbb{N}$. Into how many classes does $\equiv_n$ divide $\mathrm{Str}(L)$?

6.11 Suppose $L = \{c\}$ and $n \in \mathbb{N}$. Into how many classes does $\equiv_n$ divide $\mathrm{Str}(L)$?

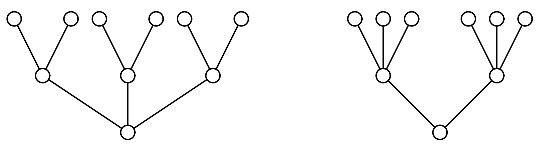
6.12 Suppose $L = \{P\}, \#(P) = 1$, and $n \in \mathbb{N}$. Into how many classes does $\equiv_n$ divide $\mathrm{Str}(L)$?

6.13 Suppose $L = \{R\}, \#(R) = 2$. Into how many classes does $\equiv_1$ divide $\mathrm{Str}(L)$?

6.14 Suppose $L = \{R\}, \#(R) = 2$. Show that $\equiv_2$ divides $\mathrm{Str}(L)$ into at least 11 classes.

6.15 Construct for each $n > 0$ trees $\mathcal{T}$ and $\mathcal{T}'$ of height 2 such that $\mathcal{T} \simeq_p^n \mathcal{T}'$ but $\mathcal{T} \not\simeq_p^{n+1} \mathcal{T}'$.

6.16 Consider $\mathrm{EF}_3(\mathcal{T}, \mathcal{T}')$ where $\mathcal{T}$ and T' are the trees below. Show that



player **I** has a winning strategy. Then write a sentence of quantifier rank 3 which is true in $\mathcal{T}$ but false in $\mathcal{T}'$.

6.17 Suppose $\mathcal{M}$ is an equivalence relation with n classes of size 1 and $n+1$ classes of size 2. Suppose, on the other hand, that $\mathcal{N}$ is an equivalence relation with $n+1$ classes of size 1 and n classes of size 2. Show that **II** has a winning strategy in $\mathrm{EF}_{n+1}(\mathcal{M}, \mathcal{M}')$ but **I** has a winning strategy in $\mathrm{EF}_{n+2}(\mathcal{M}, \mathcal{M}')$.

6.18 Suppose $\mathcal{M}$ is an equivalence relation with n classes of size k and $n+1$ classes of size $k+1$. Suppose, on the other hand, that $\mathcal{N}$ is an equivalence relation with $n+1$ classes of size k and n classes of size $k+1$. Show that **II** has a winning strategy in $\mathrm{EF}_{n+k}(\mathcal{M}, \mathcal{M}')$ but **I** has a winning strategy in $\mathrm{EF}_{n+k+1}(\mathcal{M}, \mathcal{M}')$.

6.19 Suppose $L = \{c, d\}$. Which of the following properties of L-structures $\mathcal{M}$ can be expressed in FO with a sentence of quantifier rank ≤ 1:

(a) $M \neq \{c^{\mathcal{M}}, d^{\mathcal{M}}\}$.
(b) $|M| \geq 2$.
(c) $|M \setminus \{c^{\mathcal{M}}\}| \geq 1$.
(d) $|M| = 2$.

6.20 Like Exercise 6.19 but $L = \{R\}, \#(R) = 2$, and the cases are:

(a) There is $a \in M$ such that $(b, a) \in R^{\mathcal{M}}$ for all $b \in M \setminus \{a\}$.
(b) $R^{\mathcal{M}}$ is symmetric.
(c) $R^{\mathcal{M}}$ is reflexive.

6.21 Suppose $L = \{R\}, \#(R) = 2$. Which of the following properties of L-structures $\mathcal{M}$ can be expressed in FO with a sentence of quantifier rank ≤ 2.

(a) $\mathcal{M}$ is an ordered set.
(b) $\mathcal{M}$ is a partially ordered set.
(c) $\mathcal{M}$ is an equivalence relation.
(d) $\mathcal{M}$ is a graph.

6.22 Suppose $L = \{<\}, \#(<) = 2$. Which of the following properties of L-structures $\mathcal{M}$ can be expressed in FO with a sentence of quantifier rank ≤ 3:

(a) $\mathcal{M}$ is a dense linear order.
(b) $\mathcal{M}$ is an ordered set with at least eight elements.
(c) $\mathcal{M}$ is a linear order with at least two limit points. (a is a limit point if a has predecessors but no immediate predecessor.)

6.23 Which of the following sentences are logically equivalent to a sentence of quantifier rank ≤ 1:

(a) $\forall x_0 \exists x_1 (\neg R x_0 \vee P x_1)$.
(b) $\exists x_0 \exists x_1 (R x_1 \wedge R x_0)$.
(c) $\exists x_0 \exists x_1 (\neg \approx x_0 x_1 \wedge P x_0)$.
(d) $\forall x_0 \exists x_1 \neg \approx x_0 x_1$.

6.24 Which of the following sentences are logically equivalent to a sentence of quantifier rank ≤ 2:

(a) $\forall x_0 \exists x_1 \forall x_2 (R x_0 x_2 \vee S x_0 x_1)$.
(b) $\exists x_0 \exists x_1 \forall x_2 (R x_0 x_2 \vee S x_1 x_2)$.

6.25 Suppose we are told of an ordered set $\mathcal{M}$ that $\mathcal{M} \equiv_2 (\mathbb{N}, <)$. Can we conclude that

(a) M is infinite?

(b) $\mathcal{M}$ has a smallest element?

(c) Every element has finitely many predecessors?

6.26 Show that if $\mathcal{M} \equiv_3 (\mathbb{Q}, <)$, then $\mathcal{M} \equiv (\mathbb{Q}, <)$.

6.27 Show that if $\mathcal{M} \equiv_3 (\mathbb{Z}, <)$, then $\mathcal{M} \equiv (\mathbb{Z}, <)$.

6.28 Show that for all n there are $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \equiv_n \mathcal{N}$ but $\mathcal{M} \not\equiv_{n+1} \mathcal{N}$.

6.29 Suppose L is a vocabulary and $\mathcal{A}$ an L-structure. $\mathcal{A}$ is *ω-saturated* if every type of the expanded structure $(\mathcal{A}, a_0, \ldots, a_{n-1})$, where the elements $a_0, \ldots, a_{n-1}$ are from A, is realized in $(\mathcal{A}, a_0, \ldots, a_{n-1})$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are ω-saturated structures such that $\mathcal{A} \equiv \mathcal{B}$. Show that $\mathcal{A} \simeq_p \mathcal{B}$.

6.30 Let $\mathcal{C} = \{X \in \mathcal{P}_\omega(\mathbb{R}) : \sup(X) = 1000\}$. What is a good starting move for player **I** in $G_{cub}(\mathcal{C})$? Let $\mathcal{C}' = \{X \in \mathcal{P}_\omega(\mathbb{R}) : \inf(X) \leq 1000\}$. What is a good starting move for player **II** in $G_{cub}(\mathcal{C}')$?

6.31 Let $\mathcal{C} = \{X \in \mathcal{P}_\omega(\mathbb{R}) :$ X is dense (meets every non-empty open set) in $\mathbb{R}\}$. What is a good strategy for player **II** in $G_{cub}(\mathcal{C})$?

6.32 Let $\mathcal{C} = \{X \in \mathcal{P}_\omega(\mathbb{R}) :$ every point in X is a limit point of $X\}$. What is a good strategy for player **II** in $G_{cub}(\mathcal{C})$?

6.33 Let $\mathcal{C} = \{X \subseteq \mathbb{N} : \forall m \in \mathbb{N} \exists n \in \mathbb{N}(X \cap [n, n+m] = \emptyset)\}$. What is a good strategy for player **I** in $G_{cub}(\mathcal{C})$?

6.34 Decide which player has a winning strategy in the Cub Game of the following sets:

1. $\{X \in \mathcal{P}_\omega(A) : a \in X\}$, where $a \in A$.
2. $\{X \in \mathcal{P}_\omega(A) : B \cap X$ is finite$\}$, where $B \in \mathcal{P}(A)$.
3. $\{X \in \mathcal{P}_\omega(A) : B \cap X$ is countable$\}$, where $B \in \mathcal{P}(A)$.
4. $\{X \in \mathcal{P}_\omega(\mathbb{R}) : X$ is bounded$\}$.
5. $\{X \in \mathcal{P}_\omega(\mathbb{R}) : X$ is closed$\}$.

6.35 Compute $\triangle_{a \in A} \mathcal{C}_a$ if $\mathcal{C}_a = \{X \in \mathcal{P}_\omega(A) : a \in A\}$.

6.36 Let $f : A \to A$. Let $\mathcal{C}_a = \{X \in \mathcal{P}_\omega(A) : f(a) \in X\}$ and $\mathcal{C}'_a = \{X \in \mathcal{P}_\omega(A) : f(a) \notin X\}$. Compute $\triangle_{a \in A} \mathcal{C}_a$ and $\triangledown_{a \in A} \mathcal{C}'_a$.

6.37 Suppose $\mathcal{M}$ is an ordered set. For $a \in M$ let $\mathcal{C}_a$ be the set of $X \subseteq M$ which have an element above a in $\mathcal{M}$ and let $\mathcal{C}'_a$ be the set of $X \subseteq M$ which are bounded by a in $\mathcal{M}$. Describe the sets $\triangle_{a \in M} \mathcal{C}_a$ and $\triangledown_{a \in M} \mathcal{C}_a$.

6.38 Let L be a relational vocabulary. Suppose $f : \mathcal{M} \cong \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ and L-structures such that $M = N$. If $A \subseteq M$, there is a unique submodel $\mathcal{M} \restriction A$ of $\mathcal{M}$ with domain A. Show that player **II** has a winning strategy in $G_{\text{cub}}(\{X \in \mathcal{P}_\omega(M) : \mathcal{M} \restriction A \cong \mathcal{N} \restriction A\})$.

6.39 Suppose $\mathcal{G}$ is a connected graph. Describe a winning strategy for player **II** in $G_{\text{cub}}(\mathcal{C})$, where $\mathcal{C} = \{X \in \mathcal{P}_\omega(G) : [X]_\mathcal{G} \text{ is connected}\}$.

6.40 Suppose $(A, <)$ is an ordered set and X has a last element for a stationary set of countable $X \subseteq A$. Show that $(A, <)$ itself has a last element.

6.41 Show that the set $\mathcal{CUB}_A$ of sets $\mathcal{C} \subseteq \mathcal{P}_\omega(A)$ which contain a cub is a countably closed filter (i.e. (1) If $\mathcal{C} \in \mathcal{CUB}_A$ and $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{P}_\omega(A)$, then $\mathcal{D} \in \mathcal{CUB}_A$. (2) If $\mathcal{C}_n \in \mathcal{CUB}_A$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} \mathcal{C}_n \in \mathcal{CUB}_A$). In fact, $\mathcal{CUB}_A$ is a normal filter (i.e. if $\mathcal{C}_a \in \mathcal{CUB}_A$ for all $a \in A$, then $\triangle_{a\in A}\mathcal{C}_a \in \mathcal{CUB}_A$).

6.42 Show that if $\mathcal{D}$ is stationary and $\mathcal{C}$ cub, then $\mathcal{D} \cap \mathcal{C}$ is stationary.

6.43 Show that if $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$ is stationary, then there is $n \in \mathbb{N}$ such that $\mathcal{D}_n$ is stationary.

6.44 Show that if $\mathcal{D} = \triangledown_{a\in A}\mathcal{D}_a$ is stationary, then there is $a \in A$ such that $\mathcal{D}_a$ is stationary.

6.45 (Fodor's Lemma) Suppose $\mathcal{D}$ is stationary and $f(X) \in X$ for every $X \in \mathcal{C}$. Show that there is a stationary $\mathcal{D} \subseteq \mathcal{C}$ such that f is constant on $\mathcal{D}$. (Hint: Let $\mathcal{C}_a = \{X : f(X) = a\}$. Assume no $\mathcal{C}_a$ is stationary and use Lemma 6.15 to derive a contradiction.)

6.46 Show that if A is an uncountable set, then there is a stationary set $\mathcal{C} \subseteq \mathcal{P}_\omega(A)$ such that also $\mathcal{P}_\omega(A) \setminus \mathcal{C}$ is stationary. Such sets are called *bistationary*. Note that then $\mathcal{C} \notin \mathcal{CUB}_A$. (Hint: Write $X = \{a_n^X : n \in \mathbb{N}\}$ whenever $X \in \mathcal{P}_\omega(A)$. Apply the above Fodor's Lemma to the functions $f_n(X) = a_n^X$ to find for each n a stationary $\mathcal{D}_n$ on which f_n is constant. If each $\mathcal{P}_\omega(A) \setminus \mathcal{D}_n$ is non-stationary, there is for each n a cub set $\mathcal{C}_n \subseteq \mathcal{D}_n$. Let $\mathcal{C} = \bigcap \mathcal{C}_n$ and show that $\mathcal{C}$ can have only one element, which contradicts the fact that $\mathcal{C}$ is cub.)

6.47 Use the previous exercise to conclude that $\mathcal{CUB}_A$ is not an ultrafilter (i.e. a maximal filter) if A is infinite.

6.48 Show that the set NS^A of sets $\mathcal{C} \subseteq \mathcal{P}_\omega(A)$ which are non-stationary is a σ-ideal (i.e. (1) If $\mathcal{D} \in \mathrm{NS}^A$ and $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{P}_\omega(A)$, then $\mathcal{C} \in \mathrm{NS}^A$. (2) If $\mathcal{D}_n \in \mathrm{NS}^A$ for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} \mathcal{D}_n \in \mathrm{NS}^A$). In fact, NS^A is a normal ideal (i.e. if $\mathcal{D}_a \in \mathrm{NS}^A$ for all $a \in A$, then $\triangledown_{a\in A}\mathcal{D}_a \in \mathrm{NS}^A$).

6.49 Show that if a sentence is true in a stationary set of countable submodels of a model then it is true in the model itself. More exactly: Let L be a countable vocabulary, $\mathcal{M}$ an L-model, and φ an L-sentence. Suppose $\{X \in \mathcal{P}_\omega(M) : [X]_\mathcal{M} \models \varphi\}$ is stationary. Show that $\mathcal{M} \models \varphi$.

6.50 In this and the following exercises we develop the theory of cub and stationary subsets of a regular cardinal $\kappa > \omega$. A set $C \subseteq \kappa$ is *closed* if it contains every non-zero limit ordinal $\delta < \kappa$ such that $C \cap \delta$ is unbounded in δ, and *unbounded* if it is unbounded as a subset of κ. We call $C \subseteq \kappa$

a *closed unbounded* (*cub*) set if C is both closed and unbounded. Show that the following sets are cub

(i) κ.
(ii) $\{\alpha < \kappa : \alpha \text{ is a limit ordinal}\}$.
(iii) $\{\alpha < \kappa : \alpha = \omega^\beta \text{ for some } \beta\}$.
(iv) $\{\alpha < \kappa : \text{if } \beta < \alpha \text{ and } \gamma < \alpha, \text{ then } \beta + \gamma < \alpha\}$.
(v) $\{\alpha < \kappa : \text{if } \alpha = \beta \cdot \gamma, \text{ then } \alpha = \beta \text{ or } \alpha = \gamma\}$.

6.51 Show that the following sets are not cub:

(i) $\emptyset$.
(ii) $\{\alpha < \omega_1 : \alpha = \beta + 1 \text{ for some } \beta\}$.
(iii) $\{\alpha < \omega_1 : \alpha = \omega^\beta + \omega \text{ for some } \beta\}$.
(iv) $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$.

6.52 Show that a set C contains a cub subset of ω_1 if and only if player **II** wins the game $G_\omega(W_C)$, where

$$W_C = \{(x_0, x_1, x_2, \ldots) : \sup_n x_n \in C\}.$$

6.53 A filter $\mathcal{F}$ on M is λ-*closed* if $A_\alpha \in \mathcal{F}$ for $\alpha < \beta$, where $\beta < \lambda$, implies $\bigcap_\alpha A_\alpha \in \mathcal{F}$. A filter $\mathcal{F}$ on κ is *normal* if $A_\alpha \in \mathcal{F}$ for $\alpha < \kappa$ implies $\triangle_\alpha A_\alpha \in \mathcal{F}$, where

$$\triangle_\alpha A_\alpha = \{\alpha < \kappa : \alpha \in A_\beta \text{ for all } \beta < \alpha\}.$$

Note that normality implies κ-closure. Show that if $\kappa > \omega$ is regular, then the set $\mathcal{F}$ of subsets of κ that contain a cub set is a proper normal filter on κ. The filter $\mathcal{F}$ is called the *cub-filter* on κ.

6.54 A subset of κ which meets every cub set is called *stationary*. Equivalently, a subset S of κ is stationary if its complement is not in the cub-filter. A set which is not stationary, is *non-stationary*. Show that all sets in the cub-filter are stationary. Show that

$$\{\alpha < \omega_2 : \mathrm{cof}(\alpha) = \omega\}$$

is a stationary set which is not in the cub-filter on ω_2.

6.55 (Fodor's Lemma, second formulation) Suppose $\kappa > \omega$ is a regular cardinal. If $S \subseteq \kappa$ is stationary and $f : S \to \kappa$ satisfies $f(\alpha) < \alpha$ for all $\alpha \in S$, then there is a stationary $S' \subseteq S$ such that f is constant on S'. (Hint: For each $\alpha < \kappa$ let $S_\alpha = \{\beta < \kappa : f(\beta) = \alpha\}$. Show that one of the sets S_α has to be stationary.)

6.56 Suppose κ is a regular cardinal $> \omega$. Show that there is a bistationary set $S \subseteq \kappa$ (i.e. both S and $\kappa \setminus S$ are stationary). (Hint: Note that $S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is always stationary. For $\alpha \in S$ let $\delta_\alpha : \omega \to \alpha$ be strictly increasing with $\sup_n \delta_\alpha(n) = \alpha$. By the previous exercise there is for each $n < \omega$ a stationary $A_n \subseteq S$ such that the regressive function $f_n(\alpha) = \delta_\alpha(n)$ is constant δ_n on A_n. Argue that some $\kappa \setminus A_n$ must be stationary.)

6.57 Suppose κ is a regular cardinal $> \omega$. Show that $\kappa = \bigcup_{\alpha<\kappa} S_\alpha$ where the sets S_α are disjoint stationary sets. (Hint: Proceed as in Exercise 6.56. Find $n < \omega$ such that for all $\beta < \kappa$ the set $S_\beta = \{\alpha < \kappa : \delta_\alpha(n) \geq \beta\}$ is stationary. Find stationary $S'_\beta \subseteq S_\beta$ such that $\delta_\alpha(n)$ is constant for $\alpha \in S'_\beta$. Argue that there are κ different sets S'_β.)

6.58 Show that $S \subseteq \omega_1$ is bistationary if and only if the game $G_\omega(W_S)$ is non-determined.

6.59 Suppose κ is regular $> \omega$. Show that $S \subseteq \kappa$ is stationary if and only if every regressive $f : S \to \kappa$ is constant on an unbounded set.

6.60 Prove that $C \subseteq \omega_1$ is in the cub filter if and only if almost all countable subsets of ω_1 have their sup in C.

6.61 Suppose $S \subseteq \omega_1$ is stationary. Show that for all $\alpha < \omega_1$ there is a closed subset of S of order-type $\geq \alpha$. (Hint: Prove a stronger claim by induction on α.)

6.62 Decide first which of the following are true and then show how the winner should play the game SG$(\mathcal{M}, T)$:

1. $(\mathbb{R}, <, 0) \models \exists x \forall y (y < x \vee 0 < y)$.
2. $(\mathbb{N}, <) \models \forall x \forall y (\neg y < x \vee \forall z (z < y \vee \neg z < x))$.

6.63 Prove directly that if **II** has a winning strategy in SG$(\mathcal{M}, T)$ and $\mathcal{M} \simeq_p \mathcal{N}$, then **II** has a winning strategy in SG$(\mathcal{N}, T)$.

6.64 The *Existential Semantic Game* SG$_\exists(\mathcal{M}, T)$ differs from SG$(\mathcal{M}, T)$ only in that the $\forall$-rule is omitted. Show that if **II** has a winning strategy in SG$_\exists(\mathcal{M}, T)$ and $\mathcal{M} \subseteq \mathcal{N}$, then **II** has a winning strategy in SG$_\exists(\mathcal{N}, T)$.

6.65 A formula in NNF is *existential* if it contains no universal quantifiers. (Then it is logically equivalent to one of the form $\exists x_1 \ldots \exists x_n \varphi$, where φ is quantifier free.) Show that if L is countable and T is a set of existential L-sentences, then $\mathcal{M} \models T$ if and only if player **II** has a winning strategy in the game SG$_\exists(\mathcal{M}, T)$.

6.66 The *Universal-Existential Semantic Game* SG$_{\forall\exists}(\mathcal{M}, T)$ differs from the game SG$(\mathcal{M}, T)$ only in that player **I** has to make all applications of the $\forall$-rule before all applications of the $\exists$-rule. Show that if $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq$

I	**II**
$\neg Pc \vee Pfc$	
	Pfc

Figure 6.22

... and **II** has a winning strategy in each SG$_{\forall\exists}(\mathcal{M}_n, T)$, then **II** has a winning strategy in SG$_{\forall\exists}(\cup_{n=0}^{\infty}\mathcal{M}_n, T)$.

6.67 A formula in NNF is *universal-existential* if it is of the form

$$\forall y_1 \ldots \forall y_n \exists x_1 \ldots \exists x_m \varphi,$$

where φ is quantifier free. Show that if L is countable and T is a set of universal-existential L-sentences, then $\mathcal{M} \models T$ if and only if player **II** has a winning strategy in the game SG$_{\forall\exists}(\mathcal{M}, T)$.

6.68 The *Positive Semantic Game* SG$_{\text{pos}}(\mathcal{M}, T)$ differs from SG$(\mathcal{M}, T)$ only in that the winning condition "If player **II** plays the pair (φ, s), where φ is basic, then $\mathcal{M} \models_s \varphi$" is weakened to "If player **II** plays the pair (φ, s), where φ is atomic, then $\mathcal{M} \models_s \varphi$". Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures. A surjection $h : M \to N$ is a *homomorphism* $\mathcal{M} \to \mathcal{N}$ if

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n) \Rightarrow \mathcal{N} \models \varphi(f(a_1), \ldots, f(a_n))$$

for all atomic L-formulas φ and all $a_1, \ldots, a_n \in M$. Show that if **II** has a winning strategy in SG$_{\text{pos}}(\mathcal{M}, T)$ and $h : \mathcal{M} \to \mathcal{N}$ is a surjective homomorphism, then **II** has a winning strategy in SG$_{\text{pos}}(\mathcal{N}, T)$.

6.69 A formula in NNF is *positive* if it contains no negations. Show that if L is countable and T is a set of positive L-sentences, then $\mathcal{M} \models T$ if and only if player **II** has a winning strategy in the game SG$_{\text{pos}}(\mathcal{M}, T)$.

6.70 The game MEG(T, L) is played with

$$T = \{Pc, \neg Qfc, \forall x_0(\neg Px_0 \vee Qx_0), \forall x_0(\neg Px_0 \vee Pfx_0)\}.$$

The game starts as in Figure 6.22. How does **I** play now and win?

6.71 Consider $T = \{\exists x_0 \forall x_1 Rx_0x_1, \exists x_1 \forall x_0 \neg Rx_0x_1\}$. Now we start the game MEG(T, L) as in Figure 6.23. How does **I** play now and win?

6.72 Consider $T = \{\forall x_0(\neg Px_0 \vee Qx_0), \exists x_0(Qx_0 \wedge \neg Px_0)\}$. The game MEG$(T, L)$ is played. Player **I** immediately resigns. Why?

6.73 The game MEG(T, L) is played with

$$\begin{aligned} T = \; & \{\forall x_0 \neg x_0 E x_0, \forall x_0 \forall x_1(\neg x_0 E x_1 \vee x_1 E x_0), \\ & \forall x_0 \exists x_1 x_0 E x_1, \forall x_0 \exists x_1 \neg x_0 E x_1\}. \end{aligned}$$

I	**II**
$\exists x_0 \forall x_1 R x_0 x_1$	
	$\forall x_1 R c_0 x_1$
$\exists x_1 \forall x_0 \neg R x_0 x_1$	
	$\forall x_0 \neg R x_0 c_1$

Figure 6.23

Player **I** immediately resigns. Why?

6.74 Use the game MEG(T, L) to decide whether the following sets T have a model:

1. $\{\exists x Px, \forall y(\neg Py \vee Ry)\}$.
2. $\{\forall x Pxx, \exists y \forall x \neg Pxy\}$.

6.75 Prove the following by giving a winning strategy of player **I** in the appropriate game MEG$(T \cup \{\neg\varphi\}, L)$:

1. $\{\forall x(Px \rightarrow Qx), \exists x Px\} \models \exists x Qx$.
2. $\{\forall x Rxfx\} \models \forall x \exists y Rxy$.

6.76 Suppose T is the following theory

$$
\begin{array}{l}
\forall x_0 \neg x_0 < x_0 \\
\forall x_0 \forall x_1 \forall x_2 (\neg(x_0 < x_1 \wedge x_1 < x_2) \vee x_0 < x_2) \\
\forall x_0 \forall x_1 (x_0 < x_1 \vee x_1 < x_0 \vee x_0 \approx x_1) \\
\exists x_0 (P x_0 \wedge \forall x_1 (\neg P x_1 \vee x_0 \approx x_1 \vee x_1 < x_0) \\
\exists x_0 (\neg P x_0 \wedge \forall x_1 (P x_1 \vee x_0 \approx x_1 \vee x_1 < x_0).
\end{array}
$$

Give a winning strategy for player **I** in MEG(T, L).

6.77 Prove that the relation $\sim$ is an equivalence relation on C in the proof of Lemma 6.33.

6.78 Prove that the relation $\sim$ in the proof of Lemma 6.33 has the properties: (1) If $c_i \sim c_i'$ for $1 \leq i \leq n$ and $f \in L$, then $fc_1 \ldots c_n \sim fc_1' \ldots c_n'$. (2) If $c_i \sim c_i'$ for $1 \leq i \leq n$ and $R \in L$ such that $Rc_1 \ldots c_n \in H$, then $Rc_1' \ldots c_n' \in H$.

6.79 Show in the proof of Lemma 6.33, that if $\varphi(x_1, \ldots, x_n)$ is an L-formula and $\varphi(d_1, \ldots, d_n) \in H$ for some $d_1 \ldots, d_n$, then $\mathcal{M} \models \varphi(d_1, \ldots, d_n)$.

6.80 Suppose L is a vocabulary and $\mathcal{M}$ an L-structure. Let $C = \{c_a : a \in M\}$ be a new set of constants, one for each element of M. There is a canonical expansion $\mathcal{M}^*$ of $\mathcal{M}$ to an $L \cup C$-structure where each constant c_a is interpreted as a. The *diagram* of $\mathcal{M}$ is the set $D(\mathcal{M})$ of basic $L \cup C$-sentences φ such that $\mathcal{M} \models \varphi$. Show that an L-structure $\mathcal{N}$ has a substructure isomorphic to $\mathcal{M}$ if and only if $\mathcal{N}$ can be expanded (by adding interpretations to the new constants) to a model of $D(\mathcal{M})$. You may assume M is countable although the claim is true for all M.

6.81 Show that a sentence φ is logically equivalent to an existential sentence if and only if for all $\mathcal{M} \subseteq \mathcal{N}$: If $\mathcal{M} \models \varphi$, then $\mathcal{N} \models \varphi$. (Hint: Let T be the set of existential sentences that logically imply φ. Show that a finite disjunction of sentences in T is logically equivalent to φ. Use the Compactness Theorem and the previous exercise.)

6.82 Show that a sentence φ is logically equivalent to a positive sentence if and only if for all $\mathcal{M}$ and $\mathcal{N}$: If $\mathcal{M} \models \varphi$ and $\mathcal{N}$ is a homomorphic image of $\mathcal{M}$, then $\mathcal{N} \models \varphi$.

6.83 Show that if $\mathcal{M} \equiv \mathcal{N}$, then there are $\mathcal{M}^*$ and $\mathcal{N}^*$ such that $\mathcal{M} \preceq \mathcal{M}^*$, $\mathcal{N} \preceq \mathcal{N}^*$ and $\mathcal{N}^* \cong \mathcal{M}^*$. (You may assume N and M are countable although the claim is true without this assumption.)

6.84 Suppose $\mathcal{M}$ is a structure in which $<^{\mathcal{M}}$ is a linear order of M without a last element. Show that there is $\mathcal{N}$ such that $\mathcal{M} \preceq \mathcal{N}$ and some element a of N satisfies $b <^{\mathcal{N}} a$ for all $b \in M$. (You may assume M is countable although the claim is true for all M.)

6.85 Suppose (M, R) is a partially ordered set. Prove that there is an ordered set (M, R') such that $R \subseteq R'$. (You may assume M is countable although the claim is true for all M.)

6.86 Prove using the Compactness Theorem that for every set M there is a relation $< \subseteq M \times M$ such that $(M, <)$ is an ordered set. Hint: Consider a vocabulary which has a constant symbol for each element of M. (You may assume M is countable although the claim is true for all M.)

6.87 Suppose T is a theory with an infinite model $\mathcal{M}$ in which $<^{\mathcal{M}}$ is a linear order. Show that T has a model $\mathcal{N}$ in which $<^{\mathcal{N}}$ is not well-ordered.

6.88 Suppose T is a theory which has for each $n > 0$ a model $\mathcal{M}_n$ such that $(M_n, E^{\mathcal{M}_n})$ is a graph in which there are two elements which are not connected by a path of length $\leq n$. Show that T has a model $\mathcal{N}$ in which $(N, E^{\mathcal{N}})$ is a disconnected graph.

6.89 Show that the function used in the proof of Theorem 6.38 really exists.

6.90 Suppose p is a type of T. Show that p is included in the type of some element of some model of T.

6.91 Let T be the theory of dense linear order without endpoints plus the axioms $c_n < c_m$ for natural numbers $n < m$. Show that the type $p = \{c_0 < x, c_1 < x, c_2 < x, \ldots\}$ of T is non-principal.

6.92 Suppose p and p' are types of the theory T. Does there have to be a model of T in which p is included in the type of some element and also p' is included in the type of some element? Does it make a difference if T is complete?

6.93 Suppose p and p' are types of the theory T. Under which conditions does T have a model which realizes p but omits p'? (Hint: Consider the condition: For every $\varphi(x, y)$ there is $\psi(x) \in p'$ such that for no $\delta_1(y), \ldots, \delta_n(y) \in p$ do we have both

$$T \vdash (\varphi(x, y) \wedge \delta_1(y) \wedge \ldots \wedge \delta_n(y)) \to \psi(x)$$

and

$$T \cup \{\exists x \exists y(\varphi(x, y) \wedge \delta_1(y) \wedge \ldots \wedge \delta_n(y))\} \text{ is consistent.)}$$

6.94 Let T be the theory of dense linear order without endpoints plus the axioms $c_i < c_j$ for positive and negative integers $i < j$. Let $p = \{c_0 < x, c_1 < x, c_2 < x, \ldots\}$ and $p' = \{y < c_0, y < c_{-1}, y < c_{-2}, \ldots\}$. Show that T has a model which realizes p but omits p'.

6.95 Show that if $p_0, p_1, \ldots$ are non-principal types of a countable theory T, then there is a model of T which omits each p_n.

6.96 Show that the predicate P is not explicitly definable relative to

1. $\neg\forall x Px \wedge \neg\forall x \neg Px$.
2. $\exists x \forall y((Py \wedge Qy) \to \approx xy)$.

6.97 Deduce the Craig Interpolation Theorem for arbitrary vocabularies from the assumption that it holds for relational vocabularies.

6.98 A predicate symbol occurs *positively* in a formula if the formula is in NNF and there is a non-negated occurrence of the predicate symbol in the formula. A predicate symbol occurs *negatively* in a formula if the formula is in NNF and there is a negated occurrence of the predicate symbol in the formula. Show that the Craig Interpolation Theorem holds in the following form, known as the *Lyndon Interpolation Theorem*: If L is a relational vocabulary, φ and ψ are L-sentences and $\models \varphi \to \psi$, then there is θ such that $\models \varphi \to \theta$, $\models \theta \to \psi$, every predicate symbol occurring positively in θ occurs positively in φ and ψ, and every predicate symbol symbol occurring negatively in θ occurs negatively in φ and ψ.

6.99 Assume in the previous Exercise that the sentences φ and ψ have no occurrences of the identity symbol. Assume also $\not\models \neg\varphi$ and $\not\models \psi$. Show that θ can be chosen such that it does not contain identity.

6.100 Suppose L_1 and L_2 are vocabularies which contain no function symbols. Let φ be an L_1-sentence and ψ an L_2-sentence such that ψ is universal and $\models \varphi \to \psi$. Show that there is a universal $L_1 \cap L_2$-sentence θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$.

6.101 Prove Lemma 6.44.

6.102 Prove Example 6.46.

6.103 Prove Example 6.47.

6.104 Prove Example 6.48.

6.105 Suppose L is a finite vocabulary, P_1 and P_2 are unary predicate symbols in L. Show that the class of L-structures $\mathcal{M}$ such that

$$\mathcal{M}^{(P_1^{\mathcal{M}})} \text{ and } \mathcal{M}^{(P_2^{\mathcal{M}})} \text{ are well-defined and } \mathcal{M}^{(P_1^{\mathcal{M}})} \cong \mathcal{M}^{(P_2^{\mathcal{M}})}$$

is a PC-class.

6.106 Suppose L is a finite vocabulary, P_1 and P_2 are unary predicate symbols in L. Show that for all $n \in \mathbb{N}$ the class of L-structures $\mathcal{M}$ such that

$$\mathcal{M}^{(P_1^{\mathcal{M}})} \text{ and } \mathcal{M}^{(P_2^{\mathcal{M}})} \text{ are well-defined and } \mathcal{M}^{(P_1^{\mathcal{M}})} \simeq_p^n \mathcal{M}^{(P_2^{\mathcal{M}})}$$

is a PC-class.

6.107 Suppose L is a countable vocabulary containing the binary predicate symbol $<$. Suppose T is a set of L-sentences. Prove that if T has for each $n \in \mathbb{N}$ a model $\mathcal{M}$ in which $<^{\mathcal{M}}$ is infinite or finite of length at least n, then T has a model $\mathcal{M}$ in which $<^{\mathcal{M}}$ in non-well-ordered.

6.108 Show that every PC-class is closed under isomorphisms.

6.109 Show that the intersection and union of any two PC-classes is again a PC-class.

6.110 Prove Proposition 6.51, the Tarski–Vaught Criterion.

6.111 Prove Lemma 6.55.

6.112 Prove Lemma 6.56.

6.113 Show that the Omitting Types Theorem of first-order logic fails (in its original form) for uncountable vocabularies. (Hint: Let L be a vocabulary consisting of uncountably many constants c_α and countably many constants d_n. Let T say all the constants c_α denote different elements, and all the constants d_n likewise denote different elements. Let p be the type of an element different from each d_n. Then p is non-principal in the original sense of Theorem 6.38.)

6.114 Suppose L is a vocabulary (not necessarily countable). Show that if T has a countable model, then T has a model of cardinality $2^{\aleph_0}$.

6.115 Prove that equivalence (6.16) is independent of the choice of $f_1, \ldots, f_m$.

6.116 Prove that Equation (6.17) is independent of the choice of $f_1, \ldots, f_m$.

6.117 Prove $\prod_n (\mathbb{N}, +, \cdot, 0, 1)/F \ncong (\mathbb{N}, +, \cdot, 0, 1)$, where F is a non-principal ultrafilter on $\mathbb{N}$.

6.118 Show that the ordered field $\prod_n (\mathbb{R}, <, +, \cdot, 0, 1)/F$, where F is a non-principal ultrafilter on $\mathbb{N}$, has "infinitely small" elements, i.e. elements that are greater than zero but smaller than $1/n$ for all $n \in \mathbb{N}$.

6.119 Let G_n be the graph consisting of a cycle of $n + 3$ elements, and $G = \prod_n G_n/F$. Show that G is disconnected.

7

Infinitary Logic

7.1 Introduction

As the name indicates, infinitary logic has infinite formulas. The oldest use of infinitary formulas is the elimination of quantifiers in number theory:

$$\exists x\varphi(x) \leftrightarrow \bigvee_{n\in\mathbb{N}} \varphi(n)$$

$$\forall x\varphi(x) \leftrightarrow \bigwedge_{n\in\mathbb{N}} \varphi(n).$$

Here we leave behind logic as a study of sentences humans can write down on paper. Infinitary formulas are merely mathematical objects used to study properties of structures and proofs. It turns out that games are particularly suitable for the study of infinitary logic. In a sense games replace the use of the Compactness Theorem which fails badly in infinitary logic.

7.2 Preliminary Examples

The games we have encountered so far have had a fixed length, which has been either a natural number or ω (an infinite game). Now we introduce a game which is "dynamic" in the sense that it is possible for player $\mathbf{I}$ to change the length of the game during the game. He may first claim he can win in five moves, but seeing what the first move of $\mathbf{II}$ is, he may decide he needs ten moves. In these games player $\mathbf{I}$ is not allowed to declare he will need infinitely many moves, although we shall study such games, too, later.

Before giving a rigorous definition of the Dynamic Ehrenfeucht–Fraïssé Game we discuss some simple versions of it.

Definition 7.1 (Preliminary) Suppose $\mathcal{M}, \mathcal{M}'$ are L-structures such that L is a relational vocabulary and $M \cap M' = \emptyset$. The *Dynamic Ehrenfeucht–Fraïssé Game*, denoted $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$ is defined as follows: First player **I** chooses a natural number n and then the game $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$ is played.

Note that $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$ is *not* a game of length ω. Player **II** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$ if she has one in each $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$. On the other hand, player **I** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$ if he can envisage a number n so that he has a winning strategy in $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$.

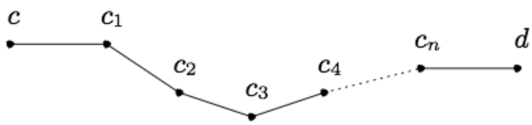
Example 7.2 If $\mathcal{M}$ and $\mathcal{M}'$ are L-structures such that M is finite and M' is infinite, then player **I** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$. Suppose $|M| = n$. Player **I** has a winning strategy in $\mathrm{EF}_{n+1}(\mathcal{M}, \mathcal{M}')$. He first plays all n elements of M and then any unplayed element of M'. Player **II** is out of good moves, and loses the game.

Example 7.3 If $\mathcal{M}$ and $\mathcal{M}'$ are equivalence relations such that $\mathcal{M}$ has finitely many equivalence classes and $\mathcal{M}'$ infinitely many, then player **I** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$. Suppose the equivalence classes of $\mathcal{M}$ are $[a_1], \ldots, [a_n]$. The strategy of **I** is to play first the elements $a_1, \ldots, a_n$. Then he plays an element from M' which is not equivalent to any element played so far. Player **II** is at a loss. She has to play an element of M equivalent to one of $a_1, \ldots, a_n$. She loses.

Definition 7.4 (Preliminary) Suppose $n \in \mathbb{N}$. The game $\mathrm{EFD}_{\omega+n}(\mathcal{M}, \mathcal{M}')$ is played as follows. First the game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ is played for n moves. Then player **I** declares a natural number m and the game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ is continued for m more moves. If **II** has not lost yet, she has won $\mathrm{EFD}_{\omega+n}(\mathcal{M}, \mathcal{M}')$. Otherwise player **I** has won.

Example 7.5 Suppose $\mathcal{G}$ and $\mathcal{G}'$ are graphs so that in $\mathcal{G}$ every vertex has a finite degree while in $\mathcal{G}'$ some vertex has infinite degree. Then player **I** has a winning strategy in $\mathrm{EFD}_{\omega+1}(\mathcal{G}, \mathcal{G}')$. Suppose $a \in G'$ has infinite degree. Player **I** plays first the element a. Let $b \in G$ be the response of player **II**. We know that every element of $\mathcal{G}$ has finite degree. Let the degree of b be n. Player **I** declares that we play $n+1$ more moves. Accordingly, he plays $n+1$ different neighbors of a. Player **II** cannot play $n+1$ different neighbors of b since b has degree n. She loses.

Example 7.6 Suppose $\mathcal{G}$ is a connected graph and $\mathcal{G}'$ a disconnected graph. Then player **I** has a winning strategy in $\mathrm{EFD}_{\omega+2}(\mathcal{G}, \mathcal{G}')$. Suppose a and b are elements of G' that are not connected by a path. Player **I** plays first elements a and b. Suppose the responses of player **II** are c and d. Since $\mathcal{G}$ is connected, there is a connected path $c = c_0, c_1, \ldots, c_n, c_{n+1} = d$ connecting c and d in $\mathcal{G}$.



Now player **I** declares that he needs n more moves. He plays the elements $c_1, \ldots, c_n$ one by one. Player **II** has to play a connected path $a_1, \ldots, a_n$ in $\mathcal{G}'$. Now d is a neighbor of c_n in $\mathcal{G}$ but b is not a neighbor of a_n in $\mathcal{G}'$ (see Figure 7.1).

Example 7.7 An *abelian group* is a structure $\mathcal{G} = (G, +)$ with $+_{\mathcal{G}} : G \times G \to G$ satisfying the conditions

(1) $x +_{\mathcal{G}} (y +_{\mathcal{G}} z) = (x +_{\mathcal{G}} y) +_{\mathcal{G}} z$ for x, y, z.
(2) there is an element $0_{\mathcal{G}}$ such that $x +_{\mathcal{G}} 0_{\mathcal{G}} = 0_{\mathcal{G}} +_{\mathcal{G}} x = x$ for all x.
(3) for all x there is $-x$ such that $x +_{\mathcal{G}} (-x) = 0_{\mathcal{G}}$.
(4) for all x and y : $x +_{\mathcal{G}} y = y +_{\mathcal{G}} x$.

Examples of abelian groups are

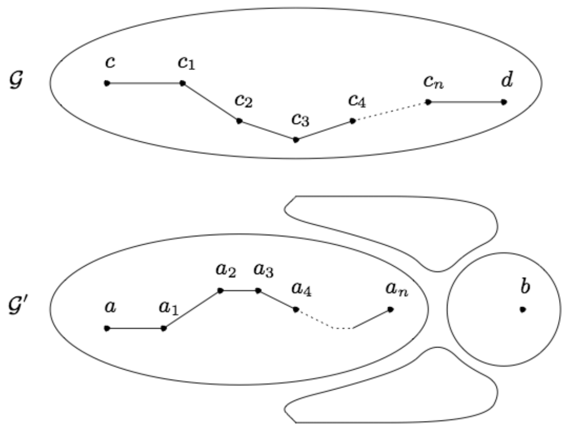


Figure 7.1

$(\mathbb{Z}, +)$	integers with addition.
$(\mathbb{Z}(n), +)$	integers modulo n with modular adddition: $x +_{\mathbb{Z}(n)} y = (x +_{\mathbb{Z}} y)$ mod n.
$(\mathbb{Q}, +)$	rationals with addition.
$(\mathbb{R}, +)$	reals with addition.
$(\mathbb{R}^+, \cdot)$	positive reals with multiplication.

Example 7.8 Consider the abelian groups $\mathbb{Z} = (\mathbb{Z}, +)$ and $\mathbb{Z}^2 = (\mathbb{Z} \times \mathbb{Z}, +)$ with

$$(m, n) + (p, q) = (m + p, n + q).$$

It is trivial that **II** has a winning strategy in $\mathrm{EFD}_1(\mathbb{Z}, \mathbb{Z}^2)$. But **I** has a winning strategy already in $\mathrm{EFD}_2(\mathbb{Z}, \mathbb{Z}^2)$: First he plays $x_0 = (1, 0)$ and $\alpha_0 = 1$. Suppose **II** responds with $y_0 \in \mathbb{Z}$. Then **I** plays $x_1 = (0, 1)$ and $\alpha_1 = 0$. Player **II** responds with $y_1 \in \mathbb{Z}$. Now

$$\sum_{i=1}^{y_1} y_0 = \sum_{i=1}^{y_0} y_1$$

but

$$\sum_{i=1}^{y_1} x_0 = (y_1, 0) \neq (0, y_0) = \sum_{i=1}^{y_0} x_1$$

unless $y_1 = y_0 = 0$, in which case **II** has lost anyway.

Example 7.9 Consider the structures $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, +, 1)$. Player **II** cannot guarantee victory even in a zero-move game, as $0 + 0 = 0$, but $1 + 1 \neq 1$. If instead we have the structures $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, \cdot, 1)$, then **II** wins the zero-move game, but if **I** has even just one move, he can play $x_0 = 0$ in $(\mathbb{Z}, \cdot, 1)$ and he wins. Namely, if **II** plays $y_0 \in \mathbb{Z}$ with $y_0 \neq 0$, we have $x_0 \cdot x_0 = x_0$ but $y_0 + y_0 \neq y_0$.

An element a of an abelian group $\mathcal{G}$ is a *torsion element* if there is an $n \in \mathbb{N}$ such that $\underbrace{a + \ldots + a}_{n} = 0$. In $\mathbb{Z}(n)$ every element is a torsion element because if $a < n$, then $\underbrace{a + \ldots + a}_{n} = na = 0$ mod n. A group in which every element is a torsion element is a *torsion group*. If no element is a torsion element, the group is *torsion-free*. Torsion-freeness can be axomatized with

$$\forall x(\underbrace{x + \ldots + x}_{n} = 0 \to x = 0), n = 1, 2, \ldots$$

Torsion groups cannot be axiomatized:

Proposition 7.10 *If T is a first-order theory in the vocabulary $\{+\}$ and $\mathbb{Z}(n) \models T$ for arbitrarily large $n \in \mathbb{N}$, then T has a model which is not a torsion group.*

Proof Let T' consist of axioms of abelian groups, T and the axioms

$$\underbrace{c + \ldots + c}_{n} \neq 0$$

for all $n \in \mathbb{N}, n > 0$. Any finite subtheory of T' is satisfied by $\mathbb{Z}(n)$ for large enough n, if we interpret c as 1. By the Compactness Theorem T' has a model $\mathcal{G}$. Let $c^{\mathcal{G}} = a$. Now in $\mathcal{G}$ we have $\underbrace{a + \ldots + a}_{n} \neq 0$ for all $n \in \mathbb{N}$. Thus a is not a torsion element of $\mathcal{G}$. □

Lemma 7.11 *If $\mathcal{G}$ is an abelian torsion group and $\mathcal{G}'$ is a non-torsion abelian group, then* **I** *has a winning strategy in* $\mathrm{EFD}_1(\mathcal{G}, \mathcal{G}')$.

Proof We let **I** play x_0 as a non-torsion element of $\mathcal{G}'$. Suppose **II** plays $y_0 \in \mathcal{G}$. Now there is $n \in \mathbb{N}$ such that

$$\underbrace{y_0 + \ldots + y_0}_{n} = 0$$

but

$$\underbrace{x_0 + \ldots + x_0}_{n} \neq 0$$

so **I** wins. □

We can construe abelian groups also as relational structures. Thus instead of a binary function $+ : G \times G \to G$ we have a ternary relation $R_+ \subseteq G \times G \times G$. Then the axioms of abelian groups are

(1) $\forall x \forall y \exists z R_+ xyz$.
(2) $\forall x \forall y \forall z \forall u ((R_+ xyz \wedge R_+ xyu) \to z = u)$.
(3) $\forall x \forall y \forall z \forall u \forall v \forall w ((R_+ xyu \wedge R_+ uzv \wedge R_+ yzw) \to R_+ xwv)$.
(4) $\exists x \forall y (R_+ xyy \wedge R_+ yxy \wedge \forall z \exists u (R_+ zux \wedge R_+ uzx))$.

In Ehrenfeucht–Fraïssé Games abelian groups behave quite differently depending on whether they are construed as relational structures or as algebraic structures.

Lemma 7.12 *If $\mathcal{G} = (G, R_+)$ is an abelian torsion group and $\mathcal{G}' = (G', R_+)$ is a non-torsion abelian group, then* **I** *has a winning strategy in the game* $\mathrm{EFD}_{\omega+1}(\mathcal{G}, \mathcal{G}')$.

Figure 7.2

Proof Let **I** play first $x_0 \in \mathcal{G}'$ which is not a torsion element. The response $y_o \in \mathcal{G}$ of **II** is a torsion element, so if we use algebraic notation, we have $z_1, \ldots, z_n$ such that

$$\begin{aligned} y_0 + y_0 &= z_1. \\ z_1 + y_0 &= z_2. \\ &\vdots \\ z_n + y_0 &= 0. \end{aligned}$$

Now **I** declares there are $n+2$ moves left, and plays $x_i = z_i$ for $i = 1, \ldots, n$. Let the responses of **II** be $y_1, \ldots, y_n$. Next **I** plays $x_{n+1} = 0_{\mathcal{G}}$, and **II** plays $y_{n+1} \in \mathcal{G}'$. Since $x_0 \in \mathcal{G}'$ is not a torsion element, **II** cannot have played $y_{n+1} = 0_{\mathcal{G}'}$ or else she loses. So there is x_{n+2} in $\mathcal{G}'$ with $x_{n+2} + y_{n+1} \neq x_{n+2}$. Now finally **I** plays this x_{n+2}, and **II** plays y_{n+2}. As $y_{n+2} + x_{n+1} = y_{n+2}$, **II** has now lost.

□

7.3 The Dynamic Ehrenfeucht–Fraïssé Game

From $\mathrm{EFD}_{\omega+n}(\mathcal{M}, \mathcal{M}')$ we could go on to define a game $\mathrm{EFD}_{\omega+\omega}(\mathcal{M}, \mathcal{M}')$ in which player **I** starts by choosing a natural number n and declaring that we are going to play the game $\mathrm{EFD}_{\omega+n}(\mathcal{M}, \mathcal{M}')$. But what is the general form of such games? We can have a situation where player **I** wants to decide that after n_0 moves he decides how many moves are left. At that point he decides that after n_1 moves he will decide how many moves now are left. At that point he decides that after n_2 moves he ... until finally he decides that the game lasts n_k more moves. A natural way of making this decision process of player **I** exact is to say that player **I** moves down an ordinal. For example, if he moves down the ordinal $\omega + \omega + 1$, he can move as in Figure 7.2.

So first he wants n_0 moves and after they have been played he decides on n_1. If he moves down on the ordinal $\omega \cdot \omega + 1$, he first chooses k and wants

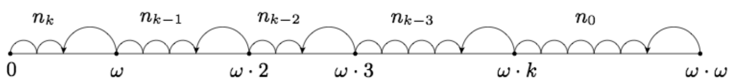


Figure 7.3

n_0 moves and after they have been played he can still make k changes of mind about the length of the rest of the game (see Figure 7.3).

Definition 7.13 Let L be a relational vocabulary and $\mathcal{M}, \mathcal{M}'$ L-structures such that $\mathcal{M} \cap \mathcal{M}' = \emptyset$. Let α be an ordinal. The Dynamic Ehrenfeucht–Fraïssé Game $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is the game $G_\omega(M \cup M' \cup \alpha, W_{\omega,\alpha}(\mathcal{M}, \mathcal{M}'))$, where $W_{\omega,\alpha}(\mathcal{M}, \mathcal{M}')$ is the set of

$$p = (x_0, \alpha_0, y_0, \ldots, x_{n-1}, \alpha_{n-1}, y_{n-1})$$

such that

(D1) For all $i < n : x_i \in M \leftrightarrow y_i \in M'$.

(D2) $\alpha > \alpha_0 > \ldots > \alpha_{n-1} = 0$.

(D3) If we denote

$$v_i = \begin{cases} x_i \text{ if } x_i \in M \\ y_i \text{ if } y_i \in M \end{cases} \quad \text{and } v'_i = \begin{cases} x_i \text{ if } x_i \in M' \\ y_i \text{ if } y_i \in M' \end{cases}$$

then

$$f_p = \{(v_0, v'_0), \cdots, (v_{n-1}, v'_{n-1})\}$$

is a partial isomorphism $\mathcal{M} \to \mathcal{M}'$.

Note that $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is *not* a game of length α. Every play in the game $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is finite, it is just how the length of the game is determined during the game where the ordinal α is used. Compared to $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$, the only new feature in $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is condition (D2). Thus $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is more difficult for **I** to play than $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$, but – if $\alpha \geq \omega$ – easier than any $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$.

Lemma 7.14 *(1) If* **II** *has a winning strategy in* $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ *and* $\beta \leq \alpha$*, then* **II** *has a winning strategy in* $\mathrm{EFD}_\beta(\mathcal{M}, \mathcal{M}')$.

(2) If **I** *has a winning strategy in* $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ *and* $\alpha \leq \beta$*, then* **I** *has a winning strategy in* $\mathrm{EFD}_\beta(\mathcal{M}, \mathcal{M}')$.

Proof (1) Any move of **I** in $\mathrm{EFD}_\beta(\mathcal{M}, \mathcal{M}')$ is as it is a legal move of **I** in $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$. Thus if **II** can beat **I** in EFD_α she can beat him in EFD_β.

(2) If **I** knows how to beat **II** in EFD_α, he can use the very same moves to beat **II** in EFD_β. □

Lemma 7.15 *If α is a limit ordinal $\neq 0$ and* **II** *has a winning strategy in the game $\mathrm{EFD}_\beta(\mathcal{M}, \mathcal{M}')$ for each $\beta < \alpha$, then* **II** *has a winning strategy in the game $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$.*

Proof In his opening move **I** plays $\alpha_0 < \alpha$. Now **II** can pretend we are actually playing the game $\mathrm{EFD}_{\alpha_0+1}(\mathcal{M}, \mathcal{M}')$. And she has a winning strategy for that game! □

Back-and-forth sequences are a way of representing a winning strategy of player **II** in the game EFD_α.

Definition 7.16 A *back-and-forth sequence* $(P_\beta : \beta \leq \alpha)$ is defined by the conditions

$$\emptyset \neq P_\alpha \subseteq \ldots \subseteq P_0 \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B}) \quad (7.1)$$

$$\forall f \in P_{\beta+1} \forall a \in A \exists b \in B \exists g \in P_\beta (f \cup \{(a, b)\} \subseteq g) \text{ for } \beta < \alpha \quad (7.2)$$

$$\forall f \in P_{\beta+1} \forall b \in B \exists a \in A \exists g \in P_\beta (f \cup \{(a, b)\} \subseteq g) \text{ for } \beta < \alpha. \quad (7.3)$$

We write

$$\mathcal{A} \simeq_p^\alpha \mathcal{B}$$

if there is a back-and-forth sequence of length α for $\mathcal{A}$ and $\mathcal{B}$.

The following proposition shows that back-and-forth sequences indeed capture the winning strategies of player **II** in $\mathrm{EFD}_\alpha(\mathcal{A}, \mathcal{B})$:

Proposition 7.17 *Suppose L is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are two L-structures. The following are equivalent:*

1. $\mathcal{A} \cong_p^\alpha \mathcal{B}$.
2. **II** *has a winning strategy in* $\mathrm{EFD}_\alpha(\mathcal{A}, \mathcal{B})$.

Proof Let us assume $A \cap B = \emptyset$. Let $(P_i : i \leq \alpha)$ be a back-and-forth sequence for $\mathcal{A}$ and $\mathcal{B}$. We define a winning strategy $\tau = (\tau_i : i \in \mathbb{N})$ for **II**. Suppose we have defined τ_i for $i < j$ and we want to define τ_j. Suppose player **I** has played $x_0, \alpha_0, \ldots, x_{j-1}, \alpha_{j-1}$ and player **II** has followed τ_i during round $i < j$. During the inductive construction of τ_i we took care to define also a partial isomorphism $f_i \in P_{\alpha_i}$ such that $\{v_0, \ldots, v_{i-1}\} \subseteq \mathrm{dom}(f_i)$. Now player **I** plays x_j and $\alpha_j < \alpha_{j-1}$. Note that $f_{j-1} \in P_{\alpha_j+1}$. By assumption there is $f_j \in P_{\alpha_j}$ extending f_{j-1} such that if $x_j \in A$, then $x_j \in \mathrm{dom}(f_j)$

and if $x_j \in B$, then $x_j \in \text{rng}(f_j)$. We let $\tau_j(x_0, \ldots, x_j) = f_j(x_j)$ if $x_j \in A$, and $\tau_j(x_0, \ldots, x_j) = f_j^{-1}(x_j)$ otherwise. This ends the construction of τ_j. This is a winning strategy because every f_p extends to a partial isomorphism $\mathcal{M} \to \mathcal{N}$.

For the converse, suppose $\tau = (\tau_n : n \in \mathbb{N})$ is a winning strategy of **II**. Let Q consist of all plays of $\text{EFD}_\alpha(\mathcal{A}, \mathcal{B})$ in which player **II** has used τ. Let P_β consist of all possible f_p where $p = (x_0, \alpha_0, y_0, \ldots, x_{i-1}, \alpha_{i-1}, y_{i-1})$ is a position in the game $\text{EFD}_\alpha(\mathcal{A}, \mathcal{B})$ with an extension in Q and $\alpha_{i-1} \geq \beta$. It is clear that $(P_\beta : \beta \leq \alpha)$ has the properties (7.1) and (7.2). □

We have already learnt in Lemma 7.14 that the bigger the ordinal α in $\text{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is, the harder it is for player **II** to win and eventually, in a typical case, her luck turns and player **I** starts to win. From that point on it is easier for **I** to win the bigger α is. Lemma 7.15, combined with the fact that the game is determined, tells us that there is a first ordinal where player **I** starts to win. So all the excitement concentrates around just one ordinal up to which player **II** has a winning strategy and starting from which player **I** has a winning strategy. It is clear that this ordinal tells us something important about the two models. This motivates the following:

Definition 7.18 An ordinal α such that player **II** has a winning strategy in $\text{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ and player **I** has a winning strategy in $\text{EFD}_{\alpha+1}(\mathcal{M}, \mathcal{M}')$ is called the *Scott watershed of* $\mathcal{M}$ *and* $\mathcal{M}'$.

By Lemma 7.14 the Scott watershed is uniquely determined, if it exists. In two extreme cases the Scott watershed does not exist. First, maybe **I** has a winning strategy even in $\text{EF}_0(\mathcal{M}, \mathcal{M}')$. Here $\text{Part}(\mathcal{M}, \mathcal{M}') = \emptyset$. Secondly, player **II** may have a winning strategy even in $\text{EF}_\omega(\mathcal{M}, \mathcal{M}')$, so **I** has no chance in any $\text{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$, and there is no Scott watershed. In any other case the Scott watershed exists. The bigger it is, the closer $\mathcal{M}$ and $\mathcal{M}'$ are to being isomorphic. Respectively, the smaller it is, the farther $\mathcal{M}$ and $\mathcal{M}'$ are from being isomorphic. If the watershed is so small that it is finite, the structures $\mathcal{M}$ and $\mathcal{M}'$ are not even elementary equivalent.

General problem: Given $\mathcal{M}$ and $\mathcal{M}'$, find the Scott watershed!

How far afield do we have to go to find the Scott watershed? It is very natural to try first some small ordinals. But if we try big ordinals, it would be nice to know how high we have to go. There is a simple answer given by the next proposition: If the models have infinite cardinality κ, and the Scott watershed exists, then it is $< \kappa^+$. Thus for countable models we only need to check

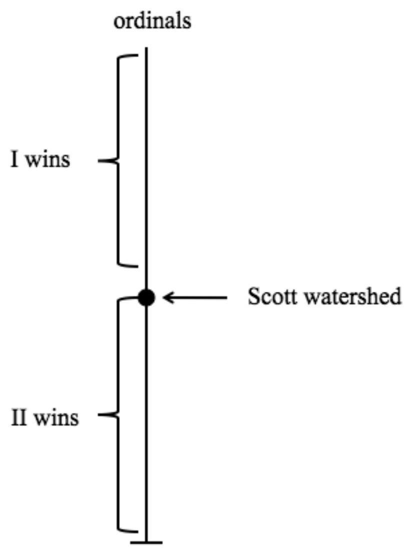


Figure 7.4

countable ordinals. For finite models this is not very interesting: if the models have at most n elements, and there is a watershed, then it is at most n.

Proposition 7.19 *If* **II** *has a winning strategy in* $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ *for all* $\alpha < (|M| + |M'|)^+$ *then* **II** *has a winning strategy in* $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$.

Proof Let $\kappa = |M| + |M'|$. The idea of **II** is to make sure that

($\star$) If the game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ has reached a position

$p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ with $f_p = \{(v_0, v'_0), \ldots, (v_{n-1}, v'_{n-1})\}$

then **II** has a winning strategy in

$$\mathrm{EFD}_{\alpha+1}((\mathcal{M}, v_0, \ldots, v_{n-1}), (\mathcal{M}', v'_0, \ldots, v'_{n-1})) \tag{7.4}$$

for all $\alpha < \kappa^+$.

In the beginning $n = 0$ and condition ($\star$) holds. Let us suppose **II** has been able to maintain ($\star$) and then **I** plays x_n in $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$. Let us look at the possibilities of **II**: She has to play some y_n and there are $\leq \kappa$ possibilities. Let Ψ be the set of them. Assume none of them works. Then for each legal move y_n there is $\alpha_{y_n} < \kappa^+$ such that

(1) **II** does not have a winning strategy in

$$\mathrm{EFD}_{\alpha_{y_n}}((\mathcal{M}, v_0, \ldots, v_n), (\mathcal{M}', v'_0, \ldots, v'_n))$$

where

$$v_n = \begin{cases} x_n \text{ if } x_n \in M \\ y_n \text{ if } x_n \in M' \end{cases} \text{ and } v'_n = \begin{cases} y_n \text{ if } y_n \in M' \\ x_n \text{ if } y_n \in M. \end{cases}$$

Let $\alpha = \sup_{y_n \in \Psi} \alpha_{y_n}$. As $|\Psi| \leq \kappa$, we have $\alpha < \kappa^+$. By the induction hypothesis, **II** has a winning strategy in the game (7.4). So, let us play this game. We let **I** play x_n and α. The winning strategy of **II** gives $y_n \in \Psi$. Let v_n and v'_n be determined as above. Now

(2) **II** has a winning strategy in $\mathrm{EFD}_\alpha((\mathcal{M}, v_0, \ldots, v_n), (\mathcal{M}', v'_0, \ldots, v'_n))$.

We have a contradiction between (1), (2), $\alpha_{y_n} < \alpha$ and Lemma 7.14. □

The above theorem is particularly important for countable models since countable partially isomorphic structures are isomorphic. Thus the countable ordinals provide a complete hierarchy of thresholds all the way from not being even elementary equivalent to being actually isomorphic. For uncountable models the hierarchy of thresholds reaches only to partial isomorphism which may be far from actual isomorphism.

We list here two structural properties of $\simeq_p^\alpha$, which are very easy to prove. There are many others and we will meet them later.

Lemma 7.20 *(Transitivity) If $\mathcal{M} \simeq_p^\alpha \mathcal{M}'$ and $\mathcal{M}' \simeq_p^\alpha \mathcal{M}''$, then $\mathcal{M} \simeq_p^\alpha \mathcal{M}''$.*

Proof Exercise 7.14. □

Lemma 7.21 *(Projection) If $\mathcal{M} \simeq_p^\alpha \mathcal{M}'$, then $\mathcal{M} \restriction L \simeq_p^\alpha \mathcal{M}' \restriction L$.*

Proof Exercise 7.15. □

We shall now introduce one of the most important concepts in infinitary logic, namely that of a Scott height of a structure. It is an invariant which sheds light on numerous aspects of the model.

Definition 7.22 The *Scott height* $\mathrm{SH}(\mathcal{M})$ of a model $\mathcal{M}$ is the supremum of all ordinals $\alpha + 1$, where α is the Scott watershed of a pair

$$(\mathcal{M}, a_1, \ldots, a_n) \not\simeq_p (\mathcal{M}, b_1, \ldots, b_n)$$

and $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$.

Lemma 7.23 $\mathrm{SH}(\mathcal{M})$ *is the least* α *such that if* $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$ *and*

$$(\mathcal{M}, a_1, \ldots, a_n) \simeq_p^{\alpha} (\mathcal{M}, b_1, \ldots, b_n)$$

then

$$(\mathcal{M}, a_1, \ldots, a_n) \simeq_p^{\alpha+1} (\mathcal{M}, b_1, \ldots, b_n).$$

Proof Exercise 7.16. □

Theorem 7.24 *If* $\mathcal{M} \simeq_p^{\mathrm{SH}(\mathcal{M})+\omega} \mathcal{M}'$*, then* $\mathcal{M} \simeq_p \mathcal{M}'$*.*

Proof Let $\mathrm{SH}(\mathcal{M}) = \alpha$. The strategy of **II** in $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ is to make sure that if the position is

$$(1) \qquad p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

then

$$(2) \qquad (\mathcal{M}, v_0, \ldots, v_{n-1}) \simeq_p^{\alpha} (\mathcal{M}', v'_0, \ldots, v'_{n-1}).$$

In the beginning of the game (2) holds by assumption. Let us then assume we are in the middle of the game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$, say in position p, and (2) holds. Now player **I** moves x_n, say $x_n = v_n \in M$. We want to find a move $y_n = v'_n \in M'$ of **II** which would yield

$$(3) \qquad (\mathcal{M}, v_0, \ldots, v_n) \simeq_p^{\alpha} (\mathcal{M}', v'_0, \ldots, v'_n).$$

Now we use the assumption $\mathcal{M} \simeq_p^{\alpha+\omega} \mathcal{M}'$. We play a sequence of rounds of an auxiliary game $G = \mathrm{EFD}_{\alpha+n+1}(\mathcal{M}, \mathcal{M}')$ in which player **II** has a winning strategy τ. First player **I** moves the elements $v'_0, \ldots, v'_{n-1}$. Let the responses of player **II** according to τ be $u_0, \ldots, u_{n-1}$. We get

$$(4) \qquad (\mathcal{M}', v'_0, \ldots, v'_{n-1}) \simeq_p^{\alpha+1} (\mathcal{M}, u_0, \ldots, u_{n-1}).$$

By transitivity,

$$(\mathcal{M}, v_0, \ldots, v_{n-1}) \simeq_p^{\alpha} (\mathcal{M}, u_0, \ldots, u_{n-1}).$$

See Figure 7.5.

By Lemma 7.23,

$$(\mathcal{M}, v_0, \ldots, v_{n-1}) \simeq_p^{\alpha+1} (\mathcal{M}, u_0, \ldots, u_{n-1}).$$

Now we apply the definition of $\simeq_p^{\alpha+1}$ and find $a \in M$ such that

$$(5) \qquad (\mathcal{M}, v_0, \ldots, v_{n-1}, v_n) \simeq_p^{\alpha} (\mathcal{M}, u_0, \ldots, u_{n-1}, a).$$

Finally we play one more round of the auxiliary game G using (4) so that

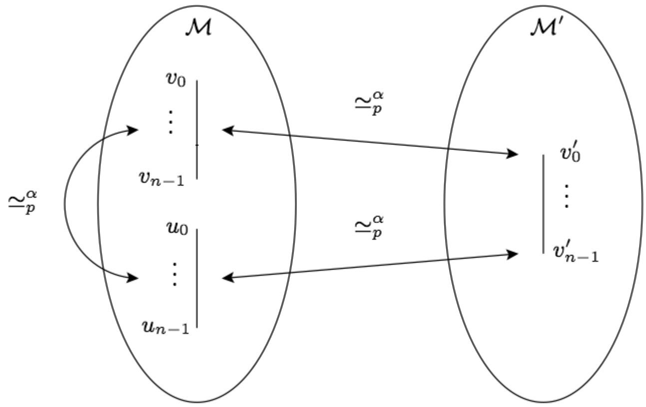


Figure 7.5

player **I** moves $a \in M$ and **II** moves according to τ an element $y_n = v'_n \in M'$. Again

$$(\mathcal{M}', v'_0, \ldots, v'_{n-1}, v'_n) \simeq_p^\alpha (\mathcal{M}, u_0, \ldots, u_{n-1}, a),$$

which together with (5) gives (3).

□

Note, that for countable models we obtain the interesting corollary:

Corollary *If $\mathcal{M}$ is countable, then for any other countable $\mathcal{M}'$ we have*

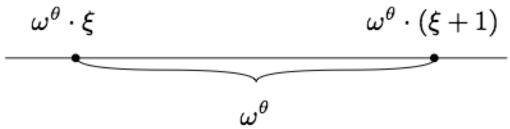
$$\mathcal{M} \simeq_p^{\mathrm{SH}(\mathcal{M})+\omega} \mathcal{M}' \iff \mathcal{M} \cong \mathcal{M}'.$$

The *Scott spectrum* $\mathrm{ss}(T)$ of a first-order theory is the class of Scott heights of its models:

$$\mathrm{ss}(T) = \{\mathrm{SH}(\mathcal{M}) : \mathcal{M} \models T\}.$$

It is in general quite difficult to determine what the Scott spectrum of a given theory is. For some theories the Scott spectrum is bounded from above. An extreme case is the case of the empty vocabulary, where the Scott height of any model is zero. It follows from Example 7.29 below that the Scott spectrum of the theory of linear order is unbounded in the class of all ordinals. A *gap* in a Scott spectrum $\mathrm{ss}(T)$ is an ordinal which is missing from $\mathrm{ss}(T)$.

Vaught's Conjecture: If T is a countable first-order theory, then T has, up to isomorphism, either $\leq \aleph_0$ or exactly $2^{\aleph_0}$ countable models.



It can be proved that any first-order theory can have only $\leq \aleph_0$ or exactly $2^{\aleph_0}$ countable models of a fixed Scott height Morley (1970). Thus, since there are $\aleph_1$ Scott heights of countable models, any first-order theory can have $\leq \aleph_1$ or exactly $2^{\aleph_0}$ countable models, up to isomorphism, all in all. To prove Vaught's Conjecture it would suffice to prove that for every first-order theory T there is an upper bound $\alpha < \omega_1$ for the Scott heights of its countable models or else there are $2^{\aleph_0}$ countable models of some fixed Scott height. This leads to the following concept: a first-order theory is *scattered* if it has at most $\aleph_0$ countable models of any fixed Scott height. Vaught's Conjecture now has the following equivalent form: *If T is scattered, then the Scott heights of its countable models have a countable upper bound.*

We now prove that there are for arbitrarily large α models with Scott height α. First we prove that for any α there are non-isomorphic models $\mathcal{M}$ and $\mathcal{M}'$ such that $\mathcal{M} \simeq_p^\alpha \mathcal{M}'$. For this we need the following useful concept:

Definition 7.25 If $\mathcal{M} = (M, <)$ and $\mathcal{M}' = (M', <')$ are ordered sets, their *product* $\mathcal{M} \times \mathcal{M}'$ is the ordered set $(M \times M', <^*)$ where

$$(x, x') <^* (y, y') \iff x' <' y' \text{ or } (x' = y' \text{ and } x < y).$$

Every ordinal α determines canonically a well-ordered set $(\alpha, <)$ which we denote also by α.

Theorem 7.26 *Suppose δ satisfies the condition*

$$\alpha < \delta \Longrightarrow \omega^\alpha < \delta$$

and $\mathcal{M}$ is any linear order with a first element. Then $\delta \simeq_p^\delta \delta \times \mathcal{M}$.

Proof An ω^θ-interval of δ is any set of the form

$$I_\xi^\theta = \{\alpha : \omega^\theta \cdot \xi \leq \alpha < \omega^\theta \cdot (\xi + 1)\}.$$

An ω^θ-interval of $\delta \times \mathcal{M}$ is any set of the form $I_\xi^\theta \times \{a\}$, where $a \in M$.

We shall define a back-and-forth sequence $P_\delta \subseteq \ldots \subseteq P_0$ as follows: A partial isomorphism f is put into P_θ if f is a finite subfunction of a partial isomorphism g from δ to $\delta \times \mathcal{M}$ such that

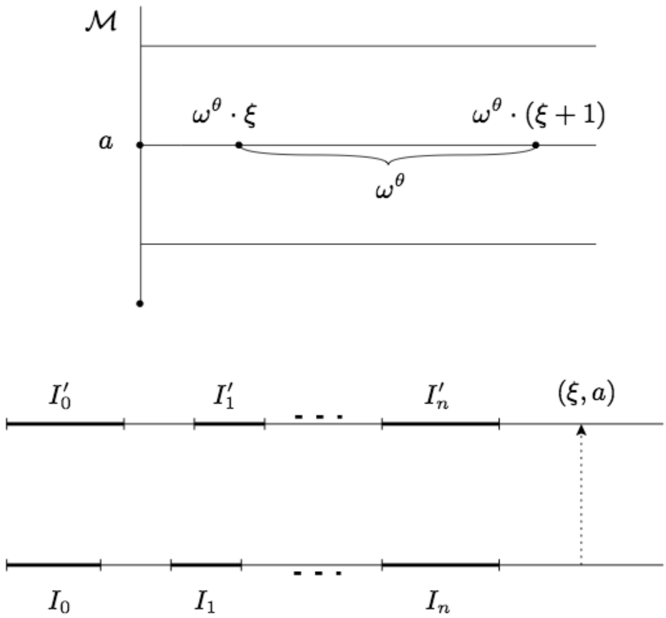


Figure 7.6

(1) $\text{dom}(g)$ is a union of finitely many ω^θ-intervals $I_0, \ldots, I_n$ of δ.
(2) $\text{rng}(g)$ is a union of finitely many ω^θ-intervals $I_0', \ldots, I_n'$ of $\delta \times \mathcal{M}$.
(3) $g(0) = (0, \min(\mathcal{M}))$.
(4) $g \restriction I_j : I_j \cong I_j'$.

The empty function is in P_δ. If $\eta < \theta$, then $P_\theta \subseteq P_\eta$ for every ω^θ-interval is a union of ω^η-intervals. To prove the back-and-forth property, suppose $f \in P_{\beta+1}$, where $\beta < \delta$ and $(\xi, a) \in \delta \times \mathcal{M}$. Suppose f is a finite subfunction of g satisfying (1)–(4). If (ξ, a) happens to be in the range of g, it is clear how to proceed: we simply extend f inside g. Let us assume that (ξ, a) is not in the range of g. Let $I_0, \ldots, I_n$ be the $\omega^{\beta+1}$-intervals in increasing order containing elements of the domain of f. Let the corresponding $\omega^{\beta+1}$-intervals in $\delta \times \mathcal{M}$ be $I_0', \ldots, I_n'$. Let m be the largest m such that (ξ, a) is above the interval I_m'. If $m = n$, we have Figure 7.6.

Let k be an isomorphism between an $\omega^{\beta+1}$-interval above I_n' and an $\omega^{\beta+1}$-interval of $\delta \times \mathcal{M}$ above I_n'. Then $g \cup k$ satisfies (1)–(4) and the restriction of $g \cup k$ to $\text{dom}(f) \cup \{k^{-1}(\xi, a)\}$ is the extension of f in P_β we are looking

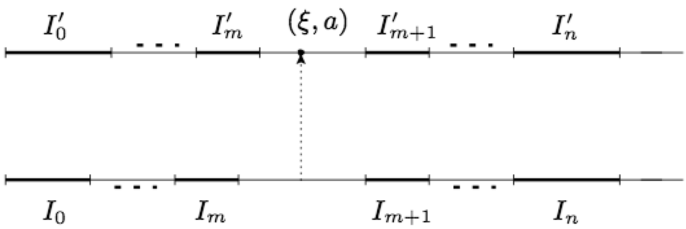


Figure 7.7

for. If on the other hand $m < n$ (Figure 7.7), we argue differently. We may not have a whole new $\omega^{\beta+1}$-interval, but we only need ω^β-intervals. So we break I_m into ω copies of ω^β-intervals $J_i (i \in \mathbb{N})$ and find a J_i which is above the finitely many elements of $\mathrm{dom}(f)$. Now we just have to choose an ω^β-interval J' containing (ξ, a) and choose an isomorphism $k : J_i \to J'$. Clearly, the restriction of $g \cup k$ to $\mathrm{dom}(f) \cup k^{-1}(\xi, a)$ is in P_β.

The other half of the back-and-forth condition is symmetric. □

Before drawing conclusions from the above important theorem we need to introduce some operations on linear orders.

The *sum* $\mathcal{M} + \mathcal{M}'$ of two linear orders $\mathcal{M}$ and $\mathcal{M}'$ is defined as the linear order consisting of $\mathcal{M}$ and $\mathcal{M}'$ one after the other, $\mathcal{M}$ first then $\mathcal{M}'$. More technically:

Definition 7.27 Suppose $\mathcal{M} = (M, <)$ and $\mathcal{M}' = (M', <')$ are linear orders. Their *sum* $\mathcal{M} + \mathcal{M}'$ is the linear order $(M'', <'')$ where

(1) $M'' = M \times \{0\} \cup M' \times \{1\}$.
(2) $(x, i) <'' (y, j) \iff i < j$ or $(i = j = 0$ and $x < y)$ or $(i = j = 1$ and $x <' y)$.

The *inverse* of a linear order $\mathcal{M} = (M, <)$ is the linear order $\mathcal{M}^* = (M, >)$. Note that if $\mathcal{M}$ is an infinite well-order, then $\mathcal{M}^*$ is necessarily non-well-ordered.

Example 7.28

$$(\mathbb{Z}, <) \cong \omega^* + \omega \ncong (\mathbb{Z}, <) + 1 + (\mathbb{Z}, <)$$
$$(\mathbb{Q}, <) \cong (\mathbb{Q}, <) + (\mathbb{Q}, <) \cong (\mathbb{Q}, <) + 1 + (\mathbb{Q}, <)$$
$$(\mathbb{R}, <) \cong (\mathbb{R}, <) + 1 + (\mathbb{R}, <) \ncong (\mathbb{R}, <) + (\mathbb{R}, <).$$

Example 7.29 Let $\alpha_0 = \omega, \alpha_{n+1} = \omega^{\alpha_n}$, and $\epsilon_0 = \sup_{n<\omega} \alpha_n$. Then

$$\epsilon_0 \simeq_p^{\epsilon_0} \epsilon_0 \times (1 + \omega^*).$$

More generally, if $\alpha = \sup_{\beta<\alpha} \omega^\beta$, then $\alpha \simeq_p^\alpha \alpha \times (1 + \omega^*)$.

The above example shows that there is *no* ordinal α such that

$$\forall \mathcal{M}((\mathcal{M} \text{ well-order } \& \mathcal{M} \simeq_p^\alpha \mathcal{M}') \to \mathcal{M}' \text{ well-order}).$$

This should be compared with the *fact*

$$\forall \mathcal{M}((\mathcal{M} \text{ well-order } \& \mathcal{M} \simeq_p \mathcal{M}') \to \mathcal{M}' \cong \mathcal{M}').$$

The above example also shows that Scott heights can be arbitrarily large and the Scott spectra of first-order theories can be unbounded in the class of all ordinals.

We now prove a result of D. Kueker about the number of automorphisms of countable models.

Lemma 7.30 *Suppose $\mathcal{M} \simeq_p \mathcal{M}'$ where $|M| < |M'|$. Then there are $a \neq a'$ in M and $b \in M'$ such that $(\mathcal{M}, a) \simeq_p (\mathcal{M}, a') \simeq_p (\mathcal{M}', b)$.*

Proof For any $b \in M'$ there is $a \in M$ such that $(\mathcal{M}, a) \simeq_p (\mathcal{M}', b)$. Since there are $|M'|$ many different b but only $|M|$ many different a, there has to be one $a_0 \in M$ such that $(\mathcal{M}, a_0) \simeq_p (\mathcal{M}', b_0)$ and $(\mathcal{M}, a_0) \simeq_p (\mathcal{M}', b_1)$ for some $b_0 \neq b_1$. Let $a_1 \in M$ such that

$$(\mathcal{M}, a_0, a_1) \simeq_p (\mathcal{M}', b_0, b_1).$$

Thus

$$(\mathcal{M}, a_0) \simeq_p (\mathcal{M}', b_1) \simeq_p (\mathcal{M}, a_1).$$

Clearly $a_0 \neq a_1$. □

Theorem 7.31 *If $\mathcal{M} \simeq_p \mathcal{M}'$ where $\mathcal{M}$ is countable and $\mathcal{M}'$ is uncountable, then $\mathcal{M}$ has $2^{\aleph_0}$ automorphisms.*

Proof We construct an automorphism π_s of $\mathcal{M}$ for each $s : \mathbb{N} \to 2$ such that if $s \neq s'$, then $\pi_s \neq \pi_{s'}$. To this end let $M = \{b_n : n \in \mathbb{N}\}$. We define π_s as the union of finite partial mappings $\pi_{s\restriction n}, n \in \mathbb{N}$. Let $\pi_\emptyset = \emptyset$. Suppose $\pi_{s\restriction n}$ has been defined and we want to define $\pi_{s\restriction n+1}$. As an induction hypothesis we assume that if

$$\pi_{s\restriction n} = \{(x_i, y_i) : i < m\}$$

then

$$\begin{array}{rcl}(\mathcal{M}, x_0, \ldots, x_{m-1}) & \simeq_p & (\mathcal{M}, y_0, \ldots, y_{m-1}) \\ & \simeq_p & (\mathcal{M}', z_0, \ldots, z_{m-1}).\end{array}$$

By Lemma 7.30 there are $a \neq a' \in M$ and $b \in M'$ such that

$$\begin{array}{rcl}(\mathcal{M}, x_0, \ldots, x_{m-1}, a) & \simeq_p & (\mathcal{M}, x_0, \ldots, x_{m-1}, a') \\ & \simeq_p & (\mathcal{M}', z_0, \ldots, z_{m-1}, b).\end{array}$$

Let $c \neq c' \in M$ such that

$$(\mathcal{M}, x_0, \ldots, x_{m-1}, a) \simeq_p (\mathcal{M}, y_0, \ldots, y_{m-1}, c)$$

and

$$(\mathcal{M}, x_0, \ldots, x_{m-1}, a, a') \simeq_p (\mathcal{M}, y_0, \ldots, y_{m-1}, c, c').$$

Then

$$\begin{array}{rcl}(\mathcal{M}, x_0, \ldots, x_{m-1}, a) & \simeq_p & (\mathcal{M}, x_0, \ldots, x_{m-1}, a') \\ & \simeq_p & (\mathcal{M}, y_0, \ldots, y_{m-1}, c').\end{array}$$

Let $x_m = a$ and

$$y_m = \begin{cases} c \text{ if } s(n) = 0 \\ c' \text{ if } s(n) = 1. \end{cases}$$

Let $c_n, d_n \in M$ such that

$$(\mathcal{M}, x_0, \ldots, x_m, b_n, c_n) \simeq_p (\mathcal{M}, y_0, \ldots, y_m, d_n, b_n)$$

and

$$\pi_{s\restriction n+1} = \{(x_i, y_i) : i \leq m\} \cup \{(b_n, d_n), (c_n, b_n)\}.$$

Two more applications of the back-and-forth property of $\simeq_p$ guarantee that the induction condition remains valid. Let

$$\pi_s = \bigcup_{n=0}^{\infty} \pi_{s\restriction n}.$$

If $s \neq s'$, say $s(n) \neq s'(n)$, then $\pi_{s\restriction n+1} \neq \pi_{s'\restriction n+1}$, so $\pi_s \neq \pi_{s'}$. Clearly each π_s is an automorphism of $\mathcal{M}$. □

Corollary *If $\mathcal{M}$ is a countable model with only countably many automorphisms, then for all $\mathcal{M}'$*

$$\mathcal{M} \simeq_p^{\mathrm{SH}(\mathcal{M})+\omega} \mathcal{M}' \iff \mathcal{M} \cong \mathcal{M}'.$$

Proof If $\mathcal{M} \simeq_p \mathcal{M}'$, then $\mathcal{M}'$ must be countable by the previous theorem. Then $\mathcal{M} \cong \mathcal{M}'$ by Proposition 5.16. □

Example 7.32 The following structures have only countably many automorphisms:

$$(\mathbb{N}, <, \cdot, 0, 1),\ (\alpha, <),\ (\mathbb{Z}, <),\ (\mathbb{Z}, +),\ (\mathbb{Q}, +).$$

7.4 Syntax and Semantics of Infinitary Logic

The syntax and semantics of the infinitary logic $L_{\infty\omega}$ that we now introduce are very much like the syntax and semantics of first-order logic. The logical symbols are $\approx, \neg, \bigwedge, \bigvee, \forall, \exists, (,), x_0, x_1, \ldots$. Terms and atomic formulas are defined as usual. Formulas of $L_{\infty\omega}$ are of the form

$$\approx tt'$$
$$Rt_1 \ldots t_n$$
$$\neg\varphi$$
$$\textstyle\bigwedge_{i\in I} \varphi_i, \bigvee_{i\in I} \varphi_i$$
$$\forall x_n\varphi, \exists x_n\varphi$$

where $t, t', t_1, \ldots, t_n$ are L-terms, $R \in L$ with $\#_l(R) = n$, and φ and all φ_i, $i \in I$, where I is an arbitrary set, are formulas of $L_{\infty\omega}$, and the formulas φ_i have altogether only finitely many free variables.[1] We regard $\varphi \wedge \psi$, $\varphi \vee \psi$, $(\varphi \to \psi)$ and $(\varphi \leftrightarrow \psi)$ as abbreviations.

In first-order logic we can think of formulas as finite strings of symbols. In infinitary logic it is customary to consider formulas as sets. Then we have the following more exact albeit more cumbersome definition:

Definition 7.33 Suppose L is a vocabulary. The class of L-formulas of $L_{\infty\omega}$ is defined as follows:

(1) If t and t' are L-terms, then $(0, t, t')$ is an L-formula denoted by $\approx tt'$.
(2) If $t_1, \ldots, t_n$ are L-terms, then $(1, R, t_1, \ldots, t_n)$ is an L-formula denoted by $Rt_1 \ldots t_n$.
(3) If φ is an L-formula, so is $(2, \varphi)$, and we denote it by $\neg\varphi$.
(4) If Φ is a set of L-formulas with a fixed finite set of free variables, then $(3, \Phi)$ is an L-formula and we denote it by $\bigwedge_{\varphi\in\Phi} \varphi$.
(5) If Φ is a set of L-formulas with a fixed finite set of free variables, then $(4, \Phi)$ is an L-formula and we denote it by $\bigvee_{\varphi\in\Phi} \varphi$.
(6) If φ is an L-formula and $n \in \mathbb{N}$, then $(5, \varphi, n)$ is an L-formula and we denote it by $\forall x_n\varphi$.

[1] This restriction makes it possible to quantify all free variables in a formula.

(7) If φ is an L-formula and $n \in \mathbb{N}$, then $(6, \varphi, n)$ is an L-formula and we denote it by $\exists x_n \varphi$.

Every formula of $L_{\infty\omega}$ is now a finite sequence of sets and the first element of the sequence is one of $\{0, 1, 2, 3, 4, 5, 6\}$. With this definition it is easy to write exact inductive definitions for various concepts related to infinitary logic.

A formula of $L_{\infty\omega}$ can be thought of as a tree, too. In this tree the formula itself is the root and the set $\mathrm{ISub}(\varphi)$ of immediate successors of a node φ of the tree are:

(1) $\mathrm{ISub}((0, t, t')) = \emptyset$.
(2) $\mathrm{ISub}((1, t_1, \ldots, t_n)) = \emptyset$.
(3) $\mathrm{ISub}((2, \varphi)) = \{\varphi\}$.
(4) $\mathrm{ISub}((3, \Phi)) = \Phi$.
(5) $\mathrm{ISub}((4, \Phi)) = \Phi$.
(6) $\mathrm{ISub}((5, \varphi, n)) = \{\varphi\}$.
(7) $\mathrm{ISub}((6, \varphi, n)) = \{\varphi\}$.

The tree thus consists of the elements of

$$\mathrm{Sub}(\varphi) = \bigcup_{n=0}^{\infty} \mathrm{Sub}_n(\varphi)$$

where

$$\begin{aligned} \mathrm{Sub}_0(\varphi) &= \{\varphi\} \\ \mathrm{Sub}_{n+1}(\varphi) &= \cup\{\mathrm{ISub}(\psi) : \psi \in \mathrm{Sub}_n(\varphi)\}, \end{aligned}$$

and the order is

$$\psi <_{Sub} \theta \iff \theta \in \mathrm{Sub}_n(\psi) \text{ for some } n > 0.$$

The tree $(\mathrm{Sub}(\varphi), <_{Sub})$ is a well-founded tree.

The quantifier rank of a formula of $L_{\infty\omega}$ is defined by induction as follows:

(1) $\mathrm{QR}(\approx tt') = 0$.
(2) $\mathrm{QR}(Rt_1 \ldots t_n) = 0$.
(3) $\mathrm{QR}(\neg\varphi) = \mathrm{QR}(\varphi)$.
(4) $\mathrm{QR}(\bigwedge \Phi) = \sup\{\mathrm{QR}(\psi) : \psi \in \Phi\}$.
(5) $\mathrm{QR}(\bigvee \Phi) = \sup\{\mathrm{QR}(\psi) : \psi \in \Phi\}$.
(6) $\mathrm{QR}(\forall x_n \varphi) = \mathrm{QR}(\varphi) + 1$.
(7) $\mathrm{QR}(\exists x_n \varphi) = \mathrm{QR}(\varphi) + 1$.

Example 7.34

$$\mathrm{QR}(\exists x_0 \ldots \exists x_n (\bigwedge_{0 \le i < j \le n} \neg{\approx} x_i x_j)) = \mathrm{QR}(\bigwedge_{0 \le i < j \le n} \neg{\approx} x_i x_j) + n + 1 = n + 1.$$

Example 7.35 Let

$$\begin{aligned} \theta_0 &= \neg \exists x_1 (x_1 < x_0) \\ \theta_\alpha &= \forall x_1 \left(x_1 < x_0 \leftrightarrow \exists x_0 \left({\approx} x_0 x_1 \wedge \left(\bigvee_{\beta < \alpha} \theta_\beta \right) \right) \right) \end{aligned}$$

All formulas θ_α are built up from two variables x_0 and x_1, and have just x_0 free. With appropriate agreements about the exchange of bound variables in substitution, these formulas could be written more succinctly as

$$\theta_\alpha(x_0) = \forall x_1 \left(x_1 < x_0 \leftrightarrow \left(\bigvee_{\beta < \alpha} \theta_\beta(x_1) \right) \right).$$

Note that

$$\mathrm{QR} \left(\forall x_1 \left(x_1 < x_0 \leftrightarrow \left(\bigvee_{\beta < \alpha} \theta_\beta(x_1) \right) \right) \right) = (\sup_{\beta < \alpha} \mathrm{QR}(\theta_\beta(x_1))) + 1.$$

Thus $\mathrm{QR}(\theta_\alpha) = \alpha + 1$.

The truth-definition of $L_{\infty\omega}$ is standard:

Definition 7.36 The concept of an assignment $s : \mathbb{N} \to M$ *satisfying* a formula φ in a model $\mathcal{M}$, $\mathcal{M} \models_s \varphi$ is defined as follows:

$$\begin{array}{lll} \mathcal{M} \models_s {\approx} t_1 t_2 & \text{iff} & t_1^{\mathcal{M}}(s) = t_2^{\mathcal{M}}(s) \\ \mathcal{M} \models_s R t_1 \ldots t_n & \text{iff} & (t_1^{\mathcal{M}}(s), \ldots, t_n^{\mathcal{M}}(s)) \in \mathrm{Val}_{\mathcal{M}}(R) \\ \mathcal{M} \models_s \neg\varphi & \text{iff} & \mathcal{M} \not\models_s \varphi \\ \mathcal{M} \models_s \bigwedge_{i \in I} \varphi_i & \text{iff} & \mathcal{M} \models_s \varphi_i \text{ for all } i \in I \\ \mathcal{M} \models_s \bigvee_{i \in I} \varphi_i & \text{iff} & \mathcal{M} \models_s \varphi_i \text{ for some } i \in I \\ \mathcal{M} \models_s \forall x_n \varphi & \text{iff} & \mathcal{M} \models_{s[a/x_n]} \varphi \text{ for all } a \in M \\ \mathcal{M} \models_s \exists x_n \varphi & \text{iff} & \mathcal{M} \models_{s[a/x_n]} \varphi \text{ for some } a \in M. \end{array}$$

An alternative definition can be given in terms of games:

Definition 7.37 Suppose L is a vocabulary, $\mathcal{M}$ is an L-structure, φ^* is an L-formula, and s^* is an assignment for M. The game $\mathrm{SG}^{\mathrm{sym}}(\mathcal{M}, \varphi^*)$ is defined as follows. In the beginning player **II** holds (φ^*, s^*). The rules of the game are as follows:

I	II
x_0	
	y_0
x_1	
	y_1
$\vdots$	$\vdots$

Figure 7.8 The game $G_\omega(W)$.

1. If φ is atomic, and s satisfies it in $\mathcal{M}$, then the player who holds (φ, s) wins the game, otherwise the other player wins.
2. If $\varphi = \neg\psi$, then the player who holds (φ, s), gives (ψ, s) to the other player.
3. If $\varphi = \bigwedge_{i \in I} \varphi_i$, then the player who holds (φ, s) switches to hold some (φ_i, s) and the other player decides which.
4. If $\varphi = \bigvee_{i \in I} \varphi_i$, then the player who holds (φ, s) switches to hold some (φ_i, s) and can himself or herself decide which.
5. If $\varphi = \forall x_n \psi$, then the player who holds (φ, s) switches to hold some $(\psi, s[a/x_n])$ and the other player chooses $a \in M$.
6. If $\varphi = \exists x_n \psi$, then the player who holds (φ, s) switches to hold some $(\psi, s[a/x_n])$ and can himself or herself choose $a \in M$.

As was pointed out in Section 6.5, $\mathcal{M} \models_s \varphi$ if and only if player **II** has a winning strategy in the above game, starting with (φ, s). Why? If $\mathcal{M} \models_s \varphi$, then the winning strategy of player **II** is to play so that if she holds (φ', s'), then $\mathcal{M} \models_{s'} \varphi'$, and if player **I** holds (φ', s'), then $\mathcal{M} \not\models_{s'} \varphi'$.

The *negation normal form* NNF is defined for $L_{\infty\omega}$ exactly as for first-order logic by requiring that negations occur in front of atomic formulas only.

Definition 7.38 The *Semantic Game* $\mathrm{SG}(\mathcal{M}, T, s)$ of the set T of L-sentences of $L_{\infty\omega}$ in NNF is the game $G_\omega(W)$(see Figure 7.8), where W consists of sequences $(x_0, y_0, x_1, y_1, \ldots)$ such that player **II** has followed the rules of Figure 7.9, and moreover, if ψ_i is a basic formula and player **II** plays the pair (ψ_i, s) then $\mathcal{M} \models_s \psi_i$.

Proposition 7.39 *$\mathcal{M} \models_s T$ iff* **II** *has a winning strategy in* $\mathrm{SG}(\mathcal{M}, T, s)$.

Proof Exercise 7.37. □

Example 7.40 Let ψ_n be the sentence $\exists x_0 \ldots \exists x_n (\bigwedge_{0 \le i < j \le n} \neg{\approx} x_i x_j)$. Then $\mathcal{M} \models (\bigvee_{n \in \mathbb{N}} \neg\psi_n)$ iff M is finite. Thus

$$\mathcal{M} \models (\bigwedge_{n \in \mathbb{N}} \psi_n) \quad \text{iff} \quad |M| \ge \aleph_0.$$

x_n	y_n	Explanation	Rule
$(\varphi, \emptyset)$		**I** enquires about $\varphi \in T$.	
	$(\varphi, \emptyset)$	**II** confirms.	Axiom rule
(φ_i, s)		**I** tests a played $(\bigwedge_{i \in I} \varphi_i, s)$ by choosing $i \in I$.	
	(φ_i, s)	**II** confirms.	$\wedge$-rule
$(\bigvee_{i \in I} \varphi_i, s)$		**I** enquires about a played disjunction.	
	(φ_i, s)	**II** makes a choice of $i \in I$.	$\vee$-rule
$(\varphi, s[a/x])$		**I** tests a played $(\forall x \varphi, s)$ by choosing $a \in M$.	
	$(\varphi, s[a/x])$	**II** confirms.	$\forall$-rule
$(\exists x \varphi, s)$		**I** enquires about a played existential statement.	
	$(\varphi, s[a/x])$	**II** makes a choice of $a \in M$.	$\exists$-rule

Figure 7.9 The game $\mathrm{SG}(\mathcal{M}, T, s)$.

Example 7.41 Let

$$\begin{aligned} \psi_0 &= \approx x_0 x_1 \\ \psi_{n+1} &= \exists x_2 (x_0 \mathrm{E} x_2 \wedge \exists x_0 (\approx x_0 x_2 \wedge \psi_n)). \end{aligned}$$

Then for graphs $\mathcal{G}$ we have

$$\mathcal{G} \models \forall x_0 \forall x_1 (\bigvee_{n \in \mathbb{N}} \psi_n) \quad \text{iff} \quad \mathcal{G} \text{ is connected.}$$

Note that the sentence $\forall x_0 \forall x_1 (\bigvee_{n \in \mathbb{N}} \psi_n)$ uses just the variables x_0, x_1, and x_2.

Example 7.42 Consider the vocabulary $\{+, 0\}$ of abelian groups. Let us introduce the notation

$$\begin{aligned} x_i \cdot 0 &= 0 \\ x_i \cdot (n+1) &= x_i \cdot n + x_i. \end{aligned}$$

Thus

$$x_i \cdot n = \underbrace{x_i + \cdots + x_i}_{n}.$$

A group $\mathcal{G}$ is torsion-free iff

$$\mathcal{G} \models \forall x_0 (\approx 0 x_0 \vee \bigwedge_{n>0} \neg\approx 0 x_0 \cdot n)$$

and $\mathcal{G}$ is torsion if

$$\mathcal{G} \models \forall x_0 (\bigvee_{n \geq 0} \approx 0 x_0 \cdot n).$$

Example 7.43 Consider the vocabulary $\{+, \cdot, 0, 1\}$ of arithmetic. Let $x_i \cdot n$ be defined as above. Then for models $\mathcal{M}$ of Peano's axioms we have

$$\mathcal{M} \cong (\mathbb{N}, +, \cdot, 0, 1) \quad \text{iff} \quad \mathcal{M} \models \forall x_0 \left(\bigvee_{n \geq 0} \approx x_0 1 \cdot n \right).$$

Example 7.44 Suppose (M, d) is a metric space. For each positive rational r let $D_r = \{(x, y) \in M \times M : d(x, y) < r\}$ and $\mathcal{M} = (M, (D_r)_{r>0})$. We can now actually define the original metric:

$$d(s(n), s(m)) = z \iff \mathcal{M} \models_s \bigwedge_{r>z>r'} (D_r x_n x_m \wedge \neg D_{r'} x_n x_m).$$

We can express the continuity of a function $f : M \to M$ with

$$(\mathcal{M}, f) \models_s \forall x_0 \left(\bigwedge_{\epsilon} \bigvee_{\delta} \forall x_1 (D_\delta x_0 x_1 \to D_\epsilon f x_0 f x_1) \right).$$

Example 7.45 Consider the formulas θ_α of Example 7.35. Then

$$\mathcal{M} \models_s \theta_\alpha \quad \text{iff} \quad (\leftarrow, s(0))^{\mathcal{M}} \cong \alpha$$

where $(\leftarrow, x)^{\mathcal{M}} = (\{y \in M : y <^{\mathcal{M}} x\}, <^{\mathcal{M}})$. We prove this by induction on α. Suppose first $f : (\leftarrow, s(0))^{\mathcal{M}} \cong \alpha$. The winning strategy of **II** in $\mathrm{SG}((\leftarrow, s(0))^{\mathcal{M}}, \theta_\alpha, s)$ is: if **I** chooses $a \in (\leftarrow, s(0))^{\mathcal{M}}$ and enquires about $\beta < \alpha$, **II** chooses $\beta = f(a)$ and plays $(\theta_\beta, s[0/a])$. By the induction hypothesis, as $(\leftarrow, a)^{\mathcal{M}} \cong \beta$, she has a winning strategy in the new position. Conversely, suppose $\mathcal{M} \models_s \theta_\alpha$. We show that $(\leftarrow, s(0))^{\mathcal{M}} \simeq_p \alpha$, from which $(\leftarrow, s(0))^{\mathcal{M}} \cong \alpha$ follows. The back-and-forth set for $(\leftarrow, s(0))^{\mathcal{M}}$ and α is the set P of finite partial isomorphisms

$$f = \{(x_0, \alpha_0), \ldots, (x_{n-1}, \alpha_{n-1})\}$$

such that for all $i < n : \mathcal{M} \models_{s[0/x_i]} \theta_{\alpha_i}$. By the induction hypothesis $(\leftarrow$

$, x_i)^{\mathcal{M}} \cong \alpha_i$. Note that isomorphisms between well-ordered sets are unique. To prove the back-and-forth property for P, suppose first $f \in P$ and $a \in (\leftarrow, s(0))^{\mathcal{M}}$. We play $\mathrm{SG}((\leftarrow, s(0))^{\mathcal{M}}, \theta_\alpha, s)$ such that player **I** enquires about $(\bigvee_{\beta<\alpha} \theta_\beta, s[0/a])$. The winning strategy of **II** yields $\beta < \alpha$ such that she plays $(\theta_\beta, s[0/a])$. By the induction hypothesis $(\leftarrow, a)^{\mathcal{M}} \cong \beta$. So $f \cup \{(a, \beta)\} \in P$. The other half of back-and-forth is proved similarly.

Example 7.46 Let θ_α be as above. Then

$$\mathcal{M} \models \left(\forall x_0 \bigvee_{\beta<\alpha} \theta_\beta\right) \wedge \left(\bigwedge_{\beta<\alpha} \exists x_0 \theta_\beta\right) \quad \text{iff} \quad \mathcal{M} \cong \alpha.$$

The proof is just as above (see Exercise 7.52).

We write $\mathcal{M} \equiv_{\infty\omega} \mathcal{M}'$ if $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same $L_{\infty\omega}$-sentences and $\mathcal{M} \equiv_\alpha \mathcal{M}'$ if they satisfy the same $L_{\infty\omega}$-sentences of quantifier rank $\leq \alpha$.

We now extend an important leg of the Strategic Balance of Logic, namely the equivalence of the Semantic Game and the Ehrenfeucht–Fraïssé Game, from first-order logic to infinitary logic:

Theorem 7.47 *The following are equivalent:*

(i) $\mathcal{A} \equiv_\alpha \mathcal{B}$.
(ii) $\mathcal{A} \simeq_p^\alpha \mathcal{B}$.

Proof $(ii) \to (i)$ Suppose $(P_\beta : \beta \leq \alpha)$ is a back-and-forth sequence for $\mathcal{A}$ and $\mathcal{B}$. We use induction on $\beta \leq \alpha$ to prove:

Claim: If $f \in P_\beta$ and $a_1, \ldots, a_k \in \mathrm{dom}(f)$, then

$$(\mathcal{A}, a_1, \ldots, a_k) \equiv_\beta (\mathcal{B}, fa_1, \ldots, fa_k).$$

We use induction on φ of quantifier rank $\leq \beta$ to prove the claim

$$(\mathcal{A}, a_1, \ldots, a_k) \models \varphi \Rightarrow (\mathcal{B}, fa_1, \ldots, fa_k) \models \varphi.$$

The only non-trivial case is that $\varphi = \exists x_n \psi(x_n)$ and $\gamma = \mathrm{QR}(\psi) < \beta$. By assumption, $f \in P_{\gamma+1}$. Since $(\mathcal{A}, a_1, \ldots, a_k) \models \varphi$, there is $a \in$ A such that $(\mathcal{A}, a_1, \ldots, a_k, a) \models \psi(c)$, where c is a new constant symbol, a name for a. Since $f \in P_{\gamma+1}$, there is $b \in B$ such that $f \cup \{(a, b)\} \in P_\gamma$. By the induction hypothesis $(\mathcal{B}, fa_1, \ldots, fa_k, b) \models \psi(c)$. Thus $(\mathcal{B}, fa_1, \ldots, fa_k) \models \varphi$.

$(i) \to (ii)$ Let P_β consist of such finite $f \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$ that if $\mathrm{dom}(f) = \{a_0, \ldots, a_{n-1}\}$, then

$$(\mathcal{A}, a_0, \ldots, a_{n-1}) \equiv_\beta (\mathcal{B}, fa_0, \ldots, fa_{n-1}).$$

By assumption (i), $\emptyset \in P_\alpha$, so $P_\alpha \neq \emptyset$. Certainly $\beta < \gamma$ implies $P_\gamma \subseteq P_\beta$. To prove the back-and-forth criterion, suppose $f \in P_{\beta+1}, a \in A$ and there is no $b \in B$ with

$$(\mathcal{A}, a_0, \ldots, a_{n-1}, a) \equiv_\beta (\mathcal{B}, fa_0, \ldots, fa_{n-1}, b). \tag{7.5}$$

Then for each $b \in B$ there is some φ_b of quantifier rank $\leq \beta$ such that

$$(\mathcal{A}, a_0, \ldots, a_{n-1}, a) \models \varphi_b(c)$$

and

$$(\mathcal{B}, fa_0, \ldots, fa_{n-1}, b) \models \neg\varphi_b(c)$$

where c is a name for a in $\mathcal{A}$ and b in $\mathcal{B}$. Thus

$$(\mathcal{A}, a_0, \ldots, a_{n-1}) \models \exists x_0 \bigwedge_{b \in B} \varphi_b(x_0).$$

Since QR$(\exists x_0 \bigwedge_{b \in B} \varphi_b(x_0)) \leq \beta + 1$ and $f \in P_{\beta+1}$ we may conclude

$$(\mathcal{B}, fa_0, \ldots, fa_{n-1}) \models \exists x_0 \bigwedge_{b \in B} \varphi_b(x_0).$$

Let $z \in B$ with $(\mathcal{B}, fa_0, \ldots, fa_{n-1}, z) \models \bigwedge_{b \in B} \varphi_b(c)$. We get the contradiction

$$(\mathcal{B}, fa_0, \ldots, fa_{n-1}, z) \models \neg\varphi_z(c) \wedge \varphi_z(c).$$

Thus $a, b \in B$ with (7.5) must exist. The other half of the back-and-forth criterion is similar. □

By combining the above theorem with our previous results about the relation $\simeq_p^\alpha$, we obtain many interesting facts about $L_{\infty\omega}$:

Proposition 7.48 *The following are equivalent for all $\mathcal{A}$ and $\mathcal{B}$:*

1. *$\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$.*
2. *$\mathcal{A} \simeq_p \mathcal{B}$ i.e. there is a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$.*
3. **II** *has a winning strategy in* $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$.

Example 7.49 (1) There is no $L_{\infty\omega}$-sentence ψ in the empty vocabulary such that for all $\mathcal{M}$:

$$\mathcal{M} \models \psi \quad \text{iff} \quad |\mathcal{M}| \leq \aleph_0,$$

because all infinite models in this vocabulary are partially isomorphic.

(2) There is no $L_{\infty\omega}$-sentence ψ in the vocabulary $\{\sim\}$ of equivalence relations such that any equivalence relation satisfies

$$\mathcal{M} \models \psi \quad \text{iff} \quad \mathcal{M} \text{ has only countably many equivalence classes.}$$

(3) There is no $L_{\infty\omega}$-sentence ψ of the vocabulary $\{E\}$ of graph theory such that for all graphs $\mathcal{G}$:

$$\mathcal{G} \models \psi \quad \text{iff} \quad \mathcal{G} \text{ has an uncountable clique.}$$

The following consequence of Karp's Theorem (Theorem 7.26) is of fundamental importance for understanding $L_{\infty\omega}$:

Corollary *There is no $L_{\infty\omega}$-sentence of the vocabulary $\{<\}$ such that for all linear orders $\mathcal{M}$:*

$$\mathcal{M} \models \psi \quad \textit{iff} \quad \mathcal{M} \textit{ is a well-order.}$$

This should be contrasted with the fact that for all α and all $\mathcal{M}$

$$\mathcal{M} \models \theta_\alpha \quad \text{iff} \quad \mathcal{M} \text{ is a well-order of type } \alpha.$$

If we could take the disjunction of all θ_α, $\alpha \in On$, we could characterize well-order, but On is a proper class, so the disjunction cannot be formed in $L_{\infty\omega}$.

Definition 7.50 Let L be a vocabulary, $\mathcal{M}$ an L-structure, and $a_0, \ldots, a_{n-1} \in M$. Then we define

$$\begin{aligned}
\sigma^0_{\mathcal{M},a_0,\ldots,a_{n-1}} &= \bigwedge\{\varphi(x_0,\ldots,x_{n-1}) : \varphi(x_0,\ldots,x_{n-1}) \\
&\qquad \text{is a basic } L\text{-formula and } \mathcal{M} \models \varphi(a_0,\ldots,a_{n-1})\} \\
\sigma^{\alpha+1}_{\mathcal{M},a_0,\ldots,a_{n-1}} &= \left(\forall x_n \bigvee_{a_n \in M} \sigma^\alpha_{\mathcal{M},a_0,\ldots,a_n}\right) \wedge \left(\bigwedge_{a_n \in M} \exists x_n \sigma^\alpha_{\mathcal{M},a_0,\ldots,a_n}\right) \\
\sigma^\nu_{\mathcal{M},a_0,\ldots,a_{n-1}} &= \bigwedge_{\alpha<\nu} \sigma^\alpha_{\mathcal{M},a_0,\ldots,a_{n-1}}, \text{ for limit } \nu \\
\sigma^\alpha_{\mathcal{M}} &= \sigma^\alpha_{\mathcal{M},\emptyset}.
\end{aligned}$$

Lemma 7.51 *1.* $\mathcal{M} \models \sigma^\alpha_{\mathcal{M},a_0,\ldots,a_{n-1}}(a_0,\ldots,a_{n-1})$.
2. $\sigma^\alpha_{\mathcal{M},a_0,\ldots,a_k}(x_0,\ldots,x_k) \models \sigma^\alpha_{\mathcal{M},a_0,\ldots,a_{n-1}}(x_0,\ldots,x_{n-1})$ *for* $n \leq k+1$.
3. If $\alpha < \beta$, *then*

$$\sigma^\beta_{\mathcal{M},a_0,\ldots,a_{n-1}}(x_0,\ldots,x_{n-1}) \models \sigma^\alpha_{\mathcal{M},a_0,\ldots,a_{n-1}}(x_0,\ldots,x_{n-1}).$$

Proof Exercise 7.44. □

Proposition 7.52 *The following are equivalent:*

(1) $\mathcal{M}' \models \sigma^\alpha_{\mathcal{M},a_0,\ldots,a_{n-1}}(b_0,\ldots,b_{n-1})$.
(2) $(\mathcal{M},a_0,\ldots,a_{n-1}) \simeq^\alpha_p (\mathcal{M}',b_0,\ldots,b_{n-1})$.

Proof Note that the quantifier rank of the formula $\sigma^\alpha_{\mathcal{M},a_0,\ldots,a_{n-1}}$ is α. Since $\mathcal{M} \models \sigma^\alpha_{\mathcal{M},a_0,\ldots,a_{n-1}}(a_0,\ldots,a_{n-1})$ by Lemma 7.51, the implication $(2) \to (1)$ follows from Proposition 7.47. Next we prove $(1) \to (2)$. Intuitively, the winning strategy of **II** in EFD_α on $(\mathcal{M},a_0,\ldots,a_{n-1})$ and $(\mathcal{M}',b_0,\ldots,b_{n-1})$ is written into the structure of $\sigma^\alpha_{\mathcal{M},a_0,\ldots,a_{n-1}}$. More exactly, we can define a back-and-forth sequence $(P_\beta : \beta \leq \alpha)$ by letting P_β consist of finite mappings

$$f = \{(a_0,b_0),\ldots,(a_{n-1},b_{n-1}),\ldots,(a_m,b_m)\}$$

such that

$$\mathcal{M}' \models \sigma^\beta_{\mathcal{M},a_0,\ldots,a_{n-1},\ldots,a_m}(b_0,\ldots,b_{n-1},\ldots,b_m)\}.$$

By the definition of the formulas $\sigma^\beta_{\mathcal{M},a_0,\ldots,a_{n-1}}$, the sequence $(P_\beta : \beta \leq \alpha)$ is indeed a back-and-forth sequence. Note that (1) implies $P_\alpha \neq \emptyset$, as $\{(a_0,b_0),\ldots,(a_{n-1},b_{n-1})\} \in P_\alpha$. □

Definition 7.53 The *Scott sentence* of a structure $\mathcal{M}$ is the $L_{\infty\omega}$-sentence

$$\sigma_\mathcal{M} = \sigma^{\mathrm{SH}(\mathcal{M})}_{\mathcal{M},\emptyset} \wedge \bigwedge_{\substack{a_0,\ldots,a_{n-1}\in M\\ n\in\mathbb{N}}} \forall x_0\ldots\forall x_{n-1}(\sigma^{\mathrm{SH}(\mathcal{M})}_{\mathcal{M},a_0,\ldots,a_{n-1}} \to \sigma^{\mathrm{SH}(\mathcal{M})+1}_{\mathcal{M},a_0,\ldots,a_{n-1}}).$$

Proposition 7.54 *The following are equivalent:*

(1) $\mathcal{M}' \models \sigma_\mathcal{M}$.
(2) $\mathcal{M}' \simeq_p \mathcal{M}$.

Proof $(2) \to (1)$: Lemma 7.51 gives $\mathcal{M} \models \sigma_\mathcal{M}$. The implication follows now from Proposition 7.48. $(1) \to (2)$: Suppose $\mathcal{M}' \models \sigma_\mathcal{M}$. We prove $\mathcal{M}' \simeq_p \mathcal{M}$ by giving a winning strategy for player **II** in the game $\mathrm{EFD}_{\mathrm{SH}(\mathcal{M})}(\mathcal{M},\mathcal{M}')$. The strategy of **II** is to make sure that if the position is

$$p = (x_0,y_0,\ldots,x_{n-1},y_{n-1})$$

then

$$(\star) \qquad \mathcal{M}' \models \sigma^{\mathrm{SH}(\mathcal{M})}_{v_0,\ldots,v_{n-1}}(v'_0,\ldots,v'_{n-1}).$$

In the beginning of the game $(\star)$ holds by assumption. Let us then assume we are in the middle of the game $\mathrm{EF}_\omega(\mathcal{M},\mathcal{M}')$, say in position p, and $(\star)$ holds. Now player **I** moves x_n, say $x_n = v_n \in M$. Now we use the assumption $\mathcal{M}' \models \sigma_\mathcal{M}$. It gives

$$\mathcal{M}' \models \sigma^{\mathrm{SH}(\mathcal{M})+1}_{\mathcal{M},v_0,\ldots,v_{n-1}}(v'_0,\ldots,v'_{n-1})),$$

whence

$$\mathcal{M}' \models \bigwedge_{a \in M} \exists y \sigma^{\alpha}_{\mathcal{M},v_0,\ldots,v_{n-1},a}(v'_0,\ldots,v'_{n-1},y).$$

By choosing $a = v_n$ we find a move $y_n = v'_n \in M'$ of **II** which yields

$$\mathcal{M}' \models \sigma^{\mathrm{SH}(\mathcal{M})}_{v_0,\ldots,v_n}(v'_0,\ldots,v'_n).$$

□

Note that if $\mathcal{M}$ is a well-ordered set then by the above result it is, up to isomorphism, the only model of $\sigma_{\mathcal{M}}$.

Corollary (Scott Isomorphism Theorem) *Suppose $\mathcal{M}$ is a countable model. Then for all countable $\mathcal{M}'$*

$$\mathcal{M}' \models \sigma_{\mathcal{M}} \iff \mathcal{M}' \cong \mathcal{M}.$$

This is a remarkable result. It puts countable models on levels of a well-ordered hierarchy according to their Scott height. On each level there is an invariant, the Scott sentence of the model, that characterizes the model up to isomorphism. These invariants need not, of course, be simple in any way, but they have a uniform tree-structure, the differences occurring only at the leaves of the tree. The invariants provide a way to systematize and classify countable models according to the syntactic properties of the Scott sentence.

For the next result we have to compute an upper bound for the number of non-equivalent infinitary formulas of a given quantifier rank. As in the finite case (see Propositions 6.3 and 4.15), the upper bound is an exponential tower, only this time we deal with infinite cardinals rather than natural numbers. These cardinal numbers look very big, but at this point the only relevant thing is that they exist. We want to be sure that there is not a proper class of non-equivalent formulas of a fixed quantifier rank. Note that if we do not limit the quantifier rank, there is a proper class of non-equivalent formulas, namely the Scott sentences of different ordinals. Recall that

$$\begin{cases} \beth_0(\lambda) = \lambda \\ \beth_{\alpha+1}(\lambda) = 2^{\beth_\alpha(\lambda)} \\ \beth_\nu(\lambda) = \sup_{\alpha<\nu} \beth_\alpha(\lambda). \end{cases}$$

Lemma 7.55 *Suppose L is a vocabulary of size μ and $\alpha = \nu + n$ where ν is a limit ordinal. There are at most $\beth_{\nu+2n+2}(\mu + \aleph_0)$ non-equivalent formulas of $L_{\infty\omega}$ of quantifier rank $\leq \alpha$.*

Proof There are at most $\mu+\aleph_0$ atomic formulas and therefore at most $\beth_2(\mu+\aleph_0)$ non-equivalent formulas of quantifier rank 0. Suppose then $\alpha = \nu+n+1$ where ν is a limit ordinal. Formulas of quantifier rank $\leq \alpha$ are of the form $\forall x_n\varphi$ or of the form $\exists x_n\varphi$, where $\mathrm{QR}(\varphi) < \alpha$, and what can be built from them by means of $\neg$, $\bigwedge_{i\in I}$ and $\bigvee_{i\in I}$. Thus their number (up to logical equivalence) is at most $\beth_2(\beth_{\nu+2n+2}(\mu+\aleph_0)) = \beth_{\nu+2(n+1)+2}(\mu+\aleph_0))$. If ν is a limit ordinal, the number of non-equivalent formulas of quantifier rank $< \nu$ is $\leq \sup_{\alpha<\nu}\beth_\alpha(\mu+\aleph_0) = \beth_\nu(\mu+\aleph_0)$. Therefore, the number of non-equivalent formulas of quantifier rank $\leq \nu$ is at most $\beth_2(\beth_\nu(\mu+\aleph_0)) = \beth_{\nu+2}(\mu+\aleph_0)$. □

Thus, for example, for any α there is only a set of non-equivalent sentences $\sigma^\alpha_{\mathcal{M}}$, while there is a proper class of non-equivalent sentences $\sigma_{\mathcal{M}}$.

Corollary *Suppose L is a vocabulary. Then for all ordinals α the equivalence relation*

$$\mathcal{A} \equiv_\alpha \mathcal{B}$$

divides the class $\mathrm{Str}(L)$ *of all L-structures into a set of equivalence classes $C^\alpha_i, i \in I$, such that if we choose any representatives $\mathcal{M}_i \in C^\alpha_i$, then:*

1. *For all L-structures $\mathcal{M}$: $\mathcal{M} \in C^\alpha_i \iff \mathcal{M} \models \sigma^\alpha_{\mathcal{M}_i}$.*
2. *If φ is an L-sentence of $L_{\infty\omega}$ of quantifier rank $\leq \alpha$, then there is a set $I_0 \subseteq I$ such that $\models \varphi \leftrightarrow \bigvee_{i\in I_0}\sigma^\alpha_{\mathcal{M}_i}$.*

Proof For any L-structure $\mathcal{M}$ let $\mathrm{Th}_\alpha(\mathcal{M})$ be the set of $L_{\infty\omega}$-sentences of quantifier rank $\leq \alpha$ (up to logical equivalence) which are true in $\mathcal{M}$. Thus

$$\mathcal{M} \equiv_\alpha \mathcal{M}' \iff \mathrm{Th}_\alpha(\mathcal{M}) = \mathrm{Th}_\alpha(\mathcal{M}').$$

Let $\mathrm{Th}_\alpha(\mathcal{M}_i), i \in I$, be a complete list of all $\mathrm{Th}_\alpha(\mathcal{M})$. The claim follows. For the second claim let I_0 consist of such $i \in I$ that $\mathcal{M}_i \models \varphi$. If $\mathcal{M} \models \varphi$ and $\mathrm{Th}_\alpha(\mathcal{M}) = \mathrm{Th}_\alpha(\mathcal{M}_i)$, then $i \in I_0$ and $\mathcal{M} \models \sigma^\alpha_{\mathcal{M}_i}$. Conversely, if $\mathcal{M} \models \sigma^\alpha_{\mathcal{M}+i}, i \in I_0$, then $\mathcal{M} \equiv_\alpha \mathcal{M}_i$ and $\mathcal{M} \models \varphi$ follows. □

Note again that if we tried to prove the above corollary for the finer relation $\simeq_p$, we would run into the difficulty that there is a proper class of equivalence classes.

Corollary *Suppose L is an arbitrary vocabulary and K is a class of L-structures. Then the following are equivalent:*

(i) *K is definable in $L_{\infty\omega}$.*
(ii) *K is closed under $\simeq^\alpha_p$ for some α.*

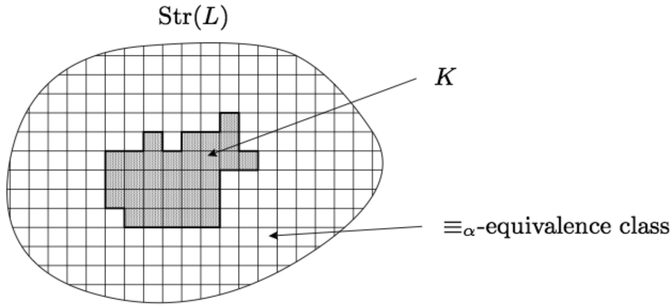


Figure 7.10 Model class K definable in $L_{\infty\omega}$.

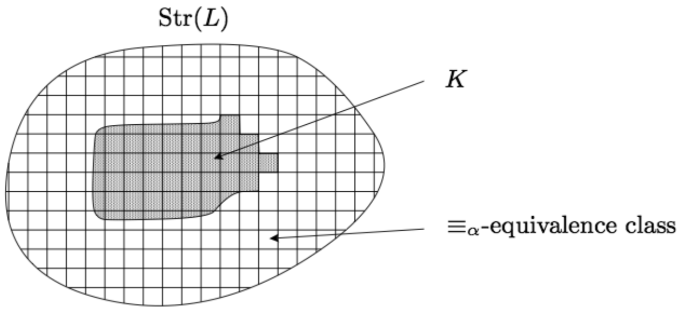


Figure 7.11 Model class K not definable in $L_{\infty\omega}$.

These equivalent conditions are strictly stronger than

(iii) *K is closed under $\simeq_p$.*

The above theorem gives a kind of normal form for sentences of $L_{\infty\omega}$: every sentence is a disjunction of sentences $\sigma^\alpha_{\mathcal{M}}$, which in turn have a very canonical form. For finite α and finite relational vocabulary the formulas $\sigma^\alpha_{\mathcal{M}}$ are first-order.

Definition 7.56 Suppose κ is a regular cardinal. $L_{\kappa\omega}$ is the fragment of $L_{\infty\omega}$ which obtains if in the definition of the syntax of $L_{\infty\omega}$ we modify condition (4) and (5) by requiring that $|I| < \kappa$.

First-order logic is in this notation $L_{\omega\omega}$. The most important non-first-order case is $L_{\omega_1\omega}$, the extension of first-order logic obtained by allowing countable disjunctions and conjunctions. Note that in a countable vocabulary the Scott sentence of a countable model is in $L_{\omega_1\omega}$.

Proposition 7.57 *Suppose $\mathcal{M}$ is a countable model in a countable vocabulary and $P \subseteq M^n$. Then the following are equivalent:*

(i) *P is closed under automorphisms of $\mathcal{M}$.*
(ii) *There is a formula $\varphi(x_0, \ldots, x_n)$ in $L_{\omega_1\omega}$ such that for all $a_0, \ldots, a_n \in M$*

$$(a_0, \ldots, a_n) \in P \iff \mathcal{M} \models \varphi(a_0, \ldots, a_n).$$

Proof $(ii) \to (i)$ is trivial because automorphisms preserve truth. To prove $(i) \to (ii)$ consider

$$\varphi(x_0, \ldots, x_n) = \bigvee\{\sigma_{(\mathcal{M},a_0,\ldots,a_n)}(x_0, \ldots, x_n) : (a_0, \ldots, a_n) \in P\},$$

where $\sigma_{(\mathcal{M},a_0,\ldots,a_n)}(x_0, \ldots, x_n)$ denotes the formula obtained from the sentence $\sigma_{(\mathcal{M},a_0,\ldots,a_n)}$ by replacing the name of a_i by the variable symbol x_i. Since M (and hence P) is countable, $\varphi(x_0, \ldots, x_n) \in L_{\omega_1\omega}$. If we now have $(a_0, \ldots, a_n) \in P$, then $\mathcal{M} \models \sigma_{(\mathcal{M},a_0,\ldots,a_n)}(a_0, \ldots, a_n)$. Thus $\mathcal{M} \models \varphi(a_0, \ldots, a_n)$. Conversely, suppose

$$\mathcal{M} \models \sigma_{(\mathcal{M},a_0,\ldots,a_n)}(b_0, \ldots, b_n) \text{ and } (a_0, \ldots, a_n) \in P.$$

Then $(\mathcal{M}, b_0, \ldots, b_n) \simeq_p (\mathcal{M}, a_0, \ldots, a_n)$. Thus there is an automorphism π of $\mathcal{M}$ such that $\pi(b_i) = a_i$ for $i \leq n$. Since P is closed under automorphisms, $(b_0, \ldots, b_n) \in P$. □

If we want to show that a relation on a countable structure is not definable in $L_{\omega_1\omega}$, a natural approach is to show that the relation is not preserved by automorphisms of the structure. The above theorem demonstrates that this natural approach is as good as any other.

Example 7.58 Let $\mathcal{M} = (\mathbb{Z}, <)$. The only subsets of $\mathbb{Z}$ that are closed under automorphisms of $\mathcal{M}$ are $\emptyset$ and $\mathbb{Z}$. Thus they are the only subsets of $\mathcal{M}$ definable in $L_{\omega_1\omega}$.

Corollary *If $\mathcal{M}$ is a rigid countable model in a countable vocabulary, then every relation on M is $L_{\omega_1\omega}$-definable on $\mathcal{M}$.*

7.5 Historical Remarks and References

Infinitary languages were introduced in propositional calculus in Scott and Tarski (1958) and in predicate logic in Tarski (1958). An early book on infinitary languages is Karp (1964). More recent books are Keisler (1971), Dickmann (1975), and Barwise (1975). A good source is the survey article Makkai (1977).

The back-and-forth sets, and thereby in effect the Ehrenfeucht–Fraïssé Game was introduced to infinitary logic in Karp (1965), where Proposition 7.48 and Proposition 7.26 appear. A good survey article on back-and-forth sets is Kueker (1975). Propositions 7.54 and 7.57 and their corollaries are from Scott (1965). Definition 7.16 is from Karp (1965). Theorem 7.31 is from Kueker (1968).

Exercises

7.1 Show that if **II** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{M},\mathcal{M}')$ and **M** is finite, then $\mathcal{M}\cong\mathcal{M}'$.

7.2 Let $\mathcal{M}=(\mathbb{Z},<)$ and $\mathcal{M}'=(\mathbb{Z}+\mathbb{Z},<)$ (i.e. two copies of $\mathcal{M}$ one after the other). For which n does **I** have a winning strategy in the game $\mathrm{EFD}_{\omega+n}(\mathcal{M},\mathcal{M}')$, and for which does **II**?

7.3 Suppose $\mathcal{G}$ and $\mathcal{G}'$ are graphs such that player **II** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{G},\mathcal{G}')$. Show that if $\mathcal{G}$ has a cycle path, then so does $\mathcal{G}'$.

7.4 Suppose $\mathcal{G}$ and $\mathcal{G}'$ are graphs and player **II** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{G},\mathcal{G}')$. Show that if $\mathcal{G}$ has infinitely many edges, also $\mathcal{G}'$ has.

7.5 Suppose $\mathcal{M}\equiv\mathcal{N}$ where $\mathcal{N}=(\mathbb{N},+,\cdot,0,1)$ but $\mathcal{M}\not\cong\mathcal{N}$. Show that **I** has a winning strategy in $\mathrm{EFD}_1(\mathcal{M},\mathcal{N})$.

7.6 Player **I** wants to play $\mathrm{EFD}_\alpha(\mathcal{M},\mathcal{M}')$ but cannot decide which α to choose. He wants to play as follows:

1. First **I** wants to play 10 moves.
2. Then, depending on how **II** has played, **I** wants to play 2^n moves for some n that he chooses.
3. Then **I** wants to play five additional moves.
4. Then, depending on how **II** has played, **I** wants to play $5n+1$ moves for some n that he chooses.
5. Finally **I** wants to play 15 additional moves, whereupon the game should end.

Can you help him choose α?

7.7 Show that player **I** (**II**) has a winning strategy in $\mathrm{EF}_n(\mathcal{M},\mathcal{M}')$ iff he (she) has a winning strategy in $\mathrm{EFD}_n(\mathcal{M},\mathcal{M}')$.

7.8 Let the game $\mathrm{EFD}^*_\alpha(\mathcal{A},\mathcal{B})$ be like the game $\mathrm{EFD}_\alpha(\mathcal{A},\mathcal{B})$ except that **I** has to play $x_{2n}\in A$ and $x_{2n+1}\in B$ for all $n\in\mathbb{N}$. Show that if ν is a limit ordinal, then player **II** has a winning strategy in $\mathrm{EFD}^*_{\nu+2n}(\mathcal{A},\mathcal{B})$ if and only if she has a winning strategy in $\mathrm{EF}_{\nu+n}(\mathcal{A},\mathcal{B})$.

7.9 Suppose $B = \{b_n : n \in \mathbb{N}\}$. Let the game $\mathrm{EFD}^{**}_\alpha(\mathcal{A}, \mathcal{B})$ be like the game $\mathrm{EFD}_\alpha(\mathcal{A}, \mathcal{B})$ except that **I** has to play $x_{2n} \in A$ and $x_{2n+1} = b_n$ for all $n \in \mathbb{N}$. Show that if ν is a limit ordinal, then player **II** has a winning strategy in $\mathrm{EFD}^{**}_{\nu+2n}(\mathcal{A}, \mathcal{B})$ if and only if she has a winning strategy in $\mathrm{EFD}_{\nu+n}(\mathcal{A}, \mathcal{B})$.

7.10 Find the Scott watershed for

(a) $(\mathbb{N}, +, \cdot, 0, 1)$ and $(\mathbb{Q}, +, \cdot, 0, 1)$.
(b) $(\mathbb{Z} + \mathbb{Z}, <)$ and $(\mathbb{Z} + \mathbb{Z} + \mathbb{Z}, <)$.

7.11 What is the Scott watershed of $(\mathbb{Z}^2, +)$ and $(\mathbb{Z}^3, +)$?

7.12 What is the Scott watershed of $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$?

7.13 Prove $\mathbb{Z}(15) \cong \mathbb{Z}(3) \times \mathbb{Z}(5)$.

7.14 Prove Lemma 7.20.

7.15 Prove Lemma 7.21.

7.16 Prove Lemma 7.23.

7.17 Show that if $(\mathcal{M}, v_0, \ldots, v_{n-1}) \simeq_p^{\mathrm{SH}(\mathcal{M})} (\mathcal{M}, v'_0, \ldots, v'_{n-1})$, then

$$(\mathcal{M}, v_0, \ldots, v_{n-1}) \simeq_p (\mathcal{M}, v'_0, \ldots, v'_{n-1}).$$

7.18 A model $\mathcal{M}$ is $\aleph_0$-*homogeneous* if the following holds for all $v_0, \ldots, v_n$ and $v'_0, \ldots, v'_{n-1}$ in M: If $(\mathcal{M}, v_0, \ldots, v_{n-1}) \equiv (\mathcal{M}, v'_0, \ldots, v'_{n-1})$ then there is v'_n in M such that $(\mathcal{M}, v_0, \ldots, v_n) \equiv (\mathcal{M}, v'_0, \ldots, v'_n)$. Show that the Scott height of an $\aleph_0$-homogeneous model is $\leq \omega$. Show that if $\mathcal{M}$ is a countable $\aleph_0$-homogeneous model and

$$(\mathcal{M}, v_0, \ldots, v_{n-1}) \equiv (\mathcal{M}, v'_0, \ldots, v'_{n-1}),$$

then there is an automorphism of $\mathcal{M}$ which maps each v_i to v'_i.

7.19 Show that there are, up to isomorphism, exactly three countable $\aleph_0$-homogeneous models $\mathcal{M}$ such that $\mathcal{M} \simeq_p^\omega (\omega, <)$.

7.20 Show that if $\mathcal{M}$ and $\mathcal{M}'$ are well-orderings, then so is $\mathcal{M} \times \mathcal{M}'$.

7.21 Prove $\alpha \cdot \beta \cong \alpha \times \beta$ starting from the inductive definition of multiplication in Exercise 2.22.

7.22 Prove $\alpha < \kappa \implies \omega^\alpha < \kappa$ for uncountable cardinals κ. Recall the inductive definition of exponentiation in Exercise 2.26.

7.23 Prove $\mathcal{M} \times (\mathcal{M}' \times \mathcal{M}'') \cong (\mathcal{M} \times \mathcal{M}') \times \mathcal{M}''$ for linear orders $\mathcal{M}, \mathcal{M}'$ and $\mathcal{M}''$. Show also that it is possible that $\mathcal{M} \times \mathcal{M}' \ncong \mathcal{M}' \times \mathcal{M}$.

7.24 Show that $\omega_1 \simeq_p^{\omega_1} \omega_1 \cdot 2$, and $(\mathbb{Q}, <) \times \omega_1 \simeq_p^{\omega_1} (\mathbb{R}, <) \times (\omega_1 \cdot 2)$.

7.25 Show that $\epsilon_0 \times (\mathbb{R}^{\geq 0}, <) \simeq_p^{\epsilon_0} \omega_1 \times (\mathbb{Q}^{\geq 0}, <)$.

7.26 Find for all α a scattered $\mathcal{M}$ and a non-scattered $\mathcal{M}'$ such that $\mathcal{M} \simeq_p^\alpha \mathcal{M}'$.

7.27 Prove the claims of Example 7.28.

7.28 Suppose $\mathcal{M} = (M, <)$ is a linear order and $\mathcal{M}'$ is the set of initial segments of $\mathcal{M}$ ordered by proper inclusion. Show that $\mathcal{M} \not\cong \mathcal{M}'$.

7.29 Show that there is a countable model $\mathcal{M}$ such that $\mathcal{M}$ has $2^{\aleph_0}$ automorphisms but there is no uncountable $\mathcal{M}'$ such that $\mathcal{M} \simeq_p \mathcal{M}'$. (Hint: Let $\mathcal{M} = (M, \omega, <, R)$ where $<$ is the usual ordering of ω, $M = \omega \cup \{(n, i) : n \in \mathbb{N}, i \in \{0, 1\}\}$ and $R = \{(n, (n, i)) : n \in \mathbb{N}, i \in \{0, 1\}\}$.)

7.30 Write a sentence of $L_{\infty\omega}$, as simple as possible, which holds in a finite graph iff

(a) the number of vertices is even.
(b) the number of edges is even.
(c) the graph has a cycle path.

7.31 Write a sentence of $L_{\infty\omega}$, as simple as possible, which holds in a finite graph iff the graph is 3-colorable. Use this to prove that there is a sentence of $L_{\infty\omega}$ which holds in a graph iff the graph is 3-colorable. (Hint: use the Compactness Theorem of propositional logic to reduce the second part to the finite case.)

7.32 An ordered field $(K, +, \cdot, 0, 1, <)$ is *Archimedian* if for all $r_1 > 0$ and r_2 in K there is a natural number n so that $\underbrace{r_1 + \cdots + r_1}_{n} > r_2$. Show that the Archimedian property can be expressed in $L_{\infty\omega}$.

7.33 Let $L^n_{\infty\omega}$ denote the fragment of $L_{\infty\omega}$ consisting of formulas in which only variables $x_0, \ldots, x_{n-1}$ occur. Show that if $\mathcal{G}$ and $\mathcal{G}'$ are graphs so that player **II** has a winning strategy in the n-Pebble Game, then they satisfy the same sentences of $L^n_{\infty\omega}$. Hence, if $\mathcal{G}$ and $\mathcal{G}'$ satisfy the extension axiom E_n, then they satisfy the same sentences of $L^n_{\infty\omega}$. (See Exercise 4.17 for the definition of the n-Pebble Game.)

7.34 Use Exercise 7.33 to conclude that if two graphs satisfy E_n for all $n \in \mathbb{N}$, then the graphs are partially isomorphic. (See Exercise 4.16 for the definition of E_n.) Conclude also that if φ is a first-order sentence, then $E_n \models \varphi$ for some $n \in \mathbb{N}$, or else $E_n \models \neg\varphi$ for some $n \in \mathbb{N}$.

7.35 Show that there is no $L_{\infty\omega}$-sentence ψ of the vocabulary of linear order such that for all linear orders $\mathcal{M}$: $\mathcal{M} \models \psi$ iff $\mathcal{M}$ has cofinality ω (i.e. has a countable unbounded subset.)

7.36 Show that there is no $L_{\infty\omega}$-sentence ψ of the vocabulary $\{P, Q\}$ of two unary predicates such that

$$(M, P^{\mathcal{M}}, Q^{\mathcal{M}}) \models \psi \quad \text{iff} \quad |P^{\mathcal{M}}| = |Q^{\mathcal{M}}|.$$

7.37 Prove Proposition 7.39.

7.38 Show that there is no $L_{\infty\omega}$-sentence ψ of the vocabulary $\{<\}$ such that for all linear orders $\mathcal{M}$:

$$\mathcal{M} \models \psi \quad \text{iff} \quad |M| \text{ is countable.}$$

Show that such a ψ exists if "linear order" is replaced by "well-order".

7.39 Let (M, d) be a metric space and $\mathcal{M} = (M, (D_r)_{r>0})$ as in Example 7.44. Write a sentence φ of $L_{\infty\omega}$ such that

(1) $(\mathcal{M}, p_0, p_1, \ldots) \models \varphi$ iff the sequence $p_0, p_1, \ldots$ converges in (M, d). (Expand the vocabulary to include names for the points p_n.)

(2) $(\mathcal{M}, f_0, f_1, \ldots, f) \models \varphi$ iff the sequence $f_0, f_1, \ldots$ of functions $f : M \to M$ converges uniformly to f. (Expand the vocabulary to include names for the functions f_n and for the function f.)

(3) $(\mathcal{M}, A) \models \varphi$ iff the set A is closed. (Expand the vocabulary to include a name for A.)

7.40 Let (M, d) and $\mathcal{M}$ be as above. Is there a sentence φ of $L_{\infty\omega}$ such that $\mathcal{M} \models \varphi$ iff (M, d) is compact?

7.41 Let V be a $\mathbb{Q}$-vector space. Let $\mathcal{M}_V = (V, +_V, 0_V, (f_r)_{r\geq 0})$, where for each non-negative rational r,

$$f_r(v) = r \cdot_V v.$$

Write a sentence φ of $L_{\infty\omega}$ such that

(1) $\mathcal{M}_V \models \varphi$ iff $\dim(V) = n$.

(2) $\mathcal{M}_V \models \varphi$ iff $\dim(V)$ is infinite.

(3) $(\mathcal{M}_V, f) \models \varphi$ iff $f : V \times V \to V$ is a linear mapping. (Expand the vocabulary to include a name for f.)

7.42 Let V be an $\mathbb{R}$-vector space with a norm $|| \cdot ||_V : V \to \mathbb{R}$. Let $\mathcal{N}_V = (V, +_V, 0_V, D_V, (f_r)_{r\geq 0})$ where $D_V = \{v \in V : ||v|| < 1\}$ and for non-negative rational r, f_r is as above. Write a sentence φ of $L_{\infty\omega}$ such that

(1) $(\mathcal{N}_V, f) \models \varphi$ iff $f : V \to V$ is continuous.

(2) $(\mathcal{N}_V, f) \models \varphi$ iff $f : V \to V$ is differentiable.

(In both (1) and (2), expand the vocabulary to include a name for f.)

7.43 Let $\mathcal{M} = (\mathbb{R}, +, \cdot, 0, 1, <)$ and L a vocabulary which extends the vocabulary of $\mathcal{M}$ by a name for a function $f : M \to M$. Write a sentence φ of $L_{\infty\omega}$ such that $(\mathcal{M}, f) \models \varphi$ iff

(1) $f \restriction [0, 1]$ has bounded variation, i.e. there exists an M such that $\sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)| \leq M$ for all $0 = x_0 < x_1 < \cdots < x_n = 1$.

(2) f is homogeneous, i.e. there is $n \in \mathbb{N}$ such that $f(ax) = a^n f(x)$ for all $a \in \mathbb{R}$.

(3) $f \restriction [0,1]$ is Riemann integrable, i.e. for each $\epsilon > 0$ there are $0 = x_0 < x_1 < \cdots < x_n = 1$ such that

$$\sum_{i=0}^{n}(x_{i+1} - x_i)\left(\sup_{x_i < x \leq x_{i+1}} f(x) - \inf_{x_i < x \leq x_{i+1}} f(x)\right) < \epsilon.$$

7.44 Prove Lemma 7.51.

7.45 Prove that if $\mathcal{M}$ is a well-ordered set, then $\mathcal{M}$ is, up to isomorphism, the only model of $\sigma_{\mathcal{M}}$.

7.46 Suppose $\mathcal{M}$ is a countable model in a countable vocabulary. Suppose $a \in M$ is fixed by all automorphisms of $\mathcal{M}$. Show that a is definable in $\mathcal{M}$ by a formula of $L_{\omega_1\omega}$.

7.47 Suppose $\mathcal{M}$ is a countable model in a countable vocabulary. Show that $\mathcal{M}$ is rigid if and only if every element of $\mathcal{M}$ is definable in $\mathcal{M}$ by a formula of $L_{\omega_1\omega}$.

7.48 Suppose $\mathcal{M}$ is a countable model in a countable vocabulary. Suppose there are $a_0, \ldots, a_{n-1} \in M$ such that $(\mathcal{M}, a_0, \ldots, a_{n-1})$ is rigid. Show that $\mathcal{M}$ can have at most countably many automorphisms.

7.49 Suppose $\mathcal{M}$ is a countable model in a countable vocabulary. Suppose $\mathcal{M}$ has $< 2^\omega$ many automorphisms. Show that there are $a_0, \ldots, a_{n-1} \in M$ such that $(\mathcal{M}, a_0, \ldots, a_{n-1})$ is rigid.

7.50 Let us write $\mathcal{M} <^\zeta_{\infty\omega} \mathcal{N}$ if $\mathcal{M} \subseteq \mathcal{N}$ and if $a_0, \ldots, a_{n-1} \in M$, then $(\mathcal{M}, a_0, \ldots, a_{n-1}) \simeq^\zeta_p (\mathcal{N}, a_0, \ldots, a_{n-1})$. Suppose $\vec{\mathcal{M}} = (\mathcal{M}_\xi : \xi < \gamma)$ is a $<^\zeta_{\infty\omega}$-chain, i.e. $\mathcal{M}_\xi <^\zeta_{\infty\omega} \mathcal{M}_\eta$ for $\xi < \eta < \gamma$. Let $\mathcal{M}$ be the union of $\vec{\mathcal{M}}$, i.e.

$$M = \bigcup_{\xi<\gamma} M_\xi,\ R^{\mathcal{M}} = \bigcup_{\xi<\gamma} R^{\mathcal{M}_\xi},\ f^{\mathcal{M}} = \bigcup_{\xi<\gamma} f^{\mathcal{M}_\xi}, c^{\mathcal{M}} = c^{\mathcal{M}_0}.$$

Show that $\mathcal{M}_\xi <^\zeta_{\infty\omega} \mathcal{M}$ for all $\xi < \gamma$.

7.51 Suppose $\mathcal{M}$ is a countable model for a countable vocabulary. Suppose there are formulas φ_n and ψ^n_m of $L_{\infty\omega}$ such that $\mathcal{M}$ satisfies the sentence:

$$\forall x_0(\textstyle\bigvee_{n<\omega} \varphi_n(x_0)) \wedge$$

$$\textstyle\bigwedge_{n<\omega} \exists x_3 \ldots \exists x_{k_n} \forall x_1(\varphi_n(x_1) \rightarrow$$
$$\textstyle\bigvee_{m<\omega} \forall x_2(\approx x_1 x_2 \leftrightarrow \psi^n_m(x_2, x_3, \ldots, x_{k_n}))).$$

Show that $\mathcal{M}$ is, up to isomorphism, the only model of $\sigma_{\mathcal{M}}$.

7.52 Prove the claim made in Example 7.46.

8

Model Theory of Infinitary Logic

8.1 Introduction

The model theory of $L_{\omega_1\omega}$ is dominated by the Model Existence Theorem. It more or less takes the role of the Compactness Theorem which can be rightfully called the cornerstone of model theory of first-order logic. The Model Existence Theorem is used to prove the Craig Interpolation Theorem and the important undefinability of the concept of well-order. When we move to the stronger logics $L_{\kappa^+\omega}$, $\kappa > \omega$, the Model Existence Theorem in general fails. However, we use a union of chains argument to prove the undefinability of well-order. In the final section we introduce game quantifiers. Here we cross the line to logics in which well-order is definable. Game quantifiers permit an approximation process which leads to the Covering Theorem, a kind of Interpolation Theorem.

8.2 Löwenheim–Skolem Theorem for $L_{\infty\omega}$

In Section 6.4 we saw that if a first-order sentence is true in a model it is true in “almost” every countable approximation of that model. We now extend this to $L_{\infty\omega}$ but of course with some modification because $L_{\infty\omega}$ has consistent sentences without any countable models. We show that if a sentence φ of $L_{\infty\omega}$ is true in a structure $\mathcal{M}$, a countable ”approximation” of φ is true in a countable ”approximation” of $\mathcal{M}$, and even more, there are this kind of approximations of φ and $\mathcal{M}$ in a sense “everywhere”. To make this statement precise we employ the Cub Game introduced in Definition 6.10. We say

$$\ldots X \ldots \text{ for almost all } X \in \mathcal{P}_\omega(A)$$

if

player **II** has a winning strategy in $G_{\text{cub}}(\mathcal{P}_\omega(A))$.

Recall the following facts:

1. If $X_0 \in \mathcal{P}_\omega(A)$, then $X_0 \subseteq X$ for almost all $X \in \mathcal{P}_\omega(A)$.
2. If $X \in \mathcal{C}$ for almost all $X \in \mathcal{P}_\omega(A)$ and $\mathcal{C} \subseteq \mathcal{C}'$, then $X \in \mathcal{C}'$ for almost all $X \in \mathcal{P}_\omega(A)$.
3. If for all $n \in \mathbb{N}$ we have $X \in \mathcal{C}_n$ for almost all $X \in \mathcal{P}_\omega(A)$, then $X \in \bigcap_{n\in\mathbb{N}} \mathcal{C}_n$ for almost all $X \in \mathcal{P}_\omega(A)$.
4. If for all $a \in A$ we have $X \in \mathcal{C}_a$ for almost all $X \in \mathcal{P}_\omega(A)$, then $X \in \triangle_{a\in A}\mathcal{C}_a$ for almost all $X \in \mathcal{P}_\omega(A)$.

In other words, the set of subsets of $\mathcal{P}_\omega(A)$ which contain almost all $X \in \mathcal{P}_\omega(A)$ is a countably complete filter.

Now that approximations extend not only to models but also to formulas we assume that models and formulas have a common universe V, which is supposed to be a transitive[1] set. As the following lemma demonstrates, the exact choice of this set V is not relevant:

Lemma 8.1 *Suppose $\emptyset \neq A \subseteq V$ and $\mathcal{C} \subseteq \mathcal{P}_\omega(A)$. Then the following are equivalent:*

1. $X \in \mathcal{C}$ for almost all $X \in \mathcal{P}_\omega(A)$.
2. $X \cap A \in \mathcal{C}$ for almost all $X \in \mathcal{P}_\omega(V)$.

Proof (1) implies (2): Let $a \in A$. Player **II** applies her winning strategy in $G_{\text{cub}}(\mathcal{C})$ in the game $G_{\text{cub}}(\{X \in \mathcal{P}_\omega(V) : X \cap A \in \mathcal{C}\})$ as follows: If **I** plays his element in A, player **II** interprets it as a move in $G_{\text{cub}}(\mathcal{C})$, where she has a winning strategy. If **I** plays x_n outside A, player **II** plays $y_n = a$. (2) implies (1): player **II** interprets all moves of **I** in A as his moves in V and then uses her winning strategy in $G_{\text{cub}}(\{X \in \mathcal{P}_\omega(V) : X \cap A \in \mathcal{C}\})$. □

Definition 8.2 Suppose $\varphi \in L_{\infty\omega}$ and X is a countable set. The approximation φ^X of φ is defined by induction as follows:

(1) $(\approx tt')^X = \approx tt'$.
(2) $(Rt_1 \ldots t_n)^X = Rt_1 \ldots t_n$.
(3) $(\neg\varphi)^X = \neg\varphi^X$.
(4) $(\bigwedge \Phi)^X = \bigwedge\{\varphi^X : \varphi \in \Phi \cap X\}$.
(5) $(\bigvee \Phi)^X = \bigvee\{\varphi^X : \varphi \in \Phi \cap X\}$.
(6) $(\forall x_n\varphi)^X = \forall x_n(\varphi^X)$.

[1] A set A is *transitive* if $y \in x \in A$ implies $y \in A$ for all x and y.

(7) $(\exists x_n\varphi)^X = \exists x_n(\varphi^X)$.

Note that φ^X is always in $L_{\omega_1\omega}$, whatever countable set X is.

Example 8.3 Suppose $X \cap \{\varphi_\alpha : \alpha < \omega_1\} = \{\varphi_{\alpha_0}, \varphi_{\alpha_1}, \ldots\}$. Then

$$\left(\forall x_0 \bigvee_{\alpha<\omega_1} \varphi_\alpha(x_0)\right)^X = \forall x_0 \bigvee_n \varphi^X_{\alpha_n}(x_0)$$

Example 8.4 Suppose $X, \mathcal{M}, \theta_\delta \in V$, V transitive, and δ is the order type of $X \cap On$. Then for all $\alpha \geq \delta$ we have $\mathcal{M} \models \forall x_0(\theta^X_\alpha \leftrightarrow \theta_\delta)$ (Exercise 8.4).

Lemma 8.5 *If $\varphi \in L_{\omega_1\omega}$, then player* **II** *has a winning strategy in the game $G_{cub}(\{X \in \mathcal{P}_\omega(V) : \varphi^X = \varphi\})$. That is, almost all approximations of $\varphi \in L_{\omega_1\omega}$ are equal to φ.*

Proof We use induction on φ. If φ is atomic, the claim is trivial since $\varphi^X = \varphi$ holds for all X. Also negation and the cases of $\forall x_n\varphi$ and $\exists x_n\varphi$ are immediate. Let us then assume $\varphi = \bigwedge_{n\in\mathbb{N}} \varphi_n$ and the claim holds for each φ_n, that is, player **II** has a winning strategy in $G_{\text{cub}}(\{X \in \mathcal{P}_\omega(V) : \varphi^X_n = \varphi_n\})$ for each n. By Lemma 6.14 player **II** has a winning strategy in the Cub Game for the set

$$\bigcap_{n\in\mathbb{N}} \{X : \varphi^X_n = \varphi_n\} \cap \{X : \varphi_n \in X \text{ for all } n \in \mathbb{N}\}.$$

□

Definition 8.6 Suppose L is a vocabulary and $\mathcal{M}$ an L-structure. Suppose φ is a first-order formula in NNF and s an assignment for the set M the domain of which includes the free variables of φ. We define the set $\mathcal{D}_{\varphi,s}$ of countable subsets of M as follows: If φ is basic, $\mathcal{D}_{\varphi,s}$ contains as an element any countable $X \subseteq V$ such that $X \cap M$ is the domain of a countable submodel $\mathcal{A}$ of $\mathcal{M}$ such that $\text{rng}(s) \subseteq A$ and:

- If φ is $\approx tt'$, then $t^{\mathcal{A}}(s) = t'^{\mathcal{A}}(t)$.
- If φ is $\neg\approx tt'$, then $t^{\mathcal{A}}(s) \neq t'^{\mathcal{A}}(t)$.
- If φ is $Rt_1 \ldots t_n$, then $(t_1^{\mathcal{A}}(s), \ldots, t_n{}^{\mathcal{A}}(t)) \in R^{\mathcal{A}}$.
- If φ is $\neg Rt_1 \ldots t_n$, then $(t_1^{\mathcal{A}}(s), \ldots, t_n{}^{\mathcal{A}}(t)) \notin R^{\mathcal{A}}$.

For non-basic φ we define

- $\mathcal{D}_{\bigwedge\Phi,s} = \triangle_{\varphi\in\Phi}\mathcal{D}_{\varphi,s}$.
- $\mathcal{D}_{\bigvee\Phi,s} = \triangledown_{\varphi\in\Phi}\mathcal{D}_{\varphi,s}$
- $\mathcal{D}_{\forall x\varphi,s} = \triangle_{a\in M}\mathcal{D}_{\varphi,s[a/x]}$.
- $\mathcal{D}_{\exists x\varphi,s} = \triangledown_{a\in M}\mathcal{D}_{\varphi,s(a/x)}$.

If φ is a sentence, we denote $\mathcal{D}_{\varphi,s}$ by $\mathcal{D}_\varphi$. If φ is not in NNF, we define $\mathcal{D}_{\varphi,s}$ and $\mathcal{D}_\varphi$ by first translating φ into a logically equivalent NNF formula.

Intuitively, $\mathcal{D}_\varphi$ is the collection of countable sets X, which *simultaneously* give an $L_{\omega_1\omega}$-approximation φ^X of φ and a countable approximation $\mathcal{M}^X$ of $\mathcal{M}$ such that $\mathcal{M}^X \models \varphi^X$.

Proposition 8.7 *Suppose $\mathcal{A}$ is an L-structure and $X \in \mathcal{D}_{\varphi,s}$. Then $[X \cap A]_\mathcal{A} \models_t \varphi^X$.*

Proof This is trivial for basic φ. For the induction step for $\bigwedge\Phi$ suppose $X \in \mathcal{D}_{\bigwedge\Phi,s}$. Suppose $\varphi \in X \cap \Phi$. Then $X \in \mathcal{D}_{\varphi,s}$. By the induction hypothesis $[X \cap A]_\mathcal{A} \models_t \varphi^X$. Thus $[X]_\mathcal{A} \models_t (\bigwedge\Phi)^X$. The other cases are as in the proof of Proposition 6.21. □

Proposition 8.8 *Suppose L is a countable vocabulary and $\mathcal{M}$ an L-structure such that $\mathcal{M} \models \varphi$. Then player* **II** *has a winning strategy in $G_{cub}(\mathcal{D}_\varphi)$.*

Proof We use induction on φ to prove that if $\mathcal{M} \models_s \varphi$, then **II** has a winning strategy in $G_{\text{cub}}(\mathcal{D}_{\varphi,s})$. Most steps are as in the proof of Proposition 6.22. Let us look at the induction step for $\bigwedge\Phi$. We assume $\mathcal{M} \models_s \bigwedge\varphi$. It suffices to prove that **II** has a winning strategy in $G_{\text{cub}}(\mathcal{D}_{\varphi,s})$ for each $\varphi \in \Phi$. But this follows from the induction hypothesis. □

Theorem 8.9 (Löwenheim–Skolem Theorem) *Suppose L is a countable vocabulary, $\mathcal{M}$ an arbitrary L-structure, and φ an $L_{\infty\omega}$-sentence of vocabulary L, and V a transitive set containing $\mathcal{M}$ and φ such that $M \cap TC(\varphi) = \emptyset$. Suppose $\mathcal{M} \models \varphi$. Let*

$$\mathcal{C} = \{X \in \mathcal{P}_\omega(V) : [X \cap M]_\mathcal{M} \models \varphi^X\}.$$

Then player **II** *has a winning strategy in the game $G_{cub}(\mathcal{C})$.*

Proof The claim follows from Propositions 8.7 and 8.8. □

Theorem 8.10 *1. $\mathcal{M} \equiv_{\infty\omega} \mathcal{N}$ if and only if $\mathcal{M}^X \cong \mathcal{N}^X$ for almost all X.*
2. $\mathcal{M} \not\equiv_{\infty\omega} \mathcal{N}$ if and only if $\mathcal{M}^X \not\cong \mathcal{N}^X$ for almost all X.

8.3 Model Theory of $L_{\omega_1\omega}$

The Model Existence Game $\text{MEG}(T, L)$ of first-order logic (Definition 6.35) can be easily modified to $L_{\omega_1\omega}$.

x_n	y_n	Explanation
φ		**I** enquires about φ.
	φ	**II** confirms.
$\approx tt$		**I** enquires about an equation.
	$\approx tt$	**II** confirms.
$\varphi(t)$		**I** chooses played $\varphi(c)$ and $\approx ct$ with φ basic and enquires about substituting t for c in φ.
	$\varphi(t)$	**II** confirms.
φ_i		**I** tests a played $\bigwedge_{i \in I} \varphi_i$ by choosing $i \in I$.
	φ_i	**II** confirms.
$\bigvee_{i \in I} \varphi_i$		**I** enquires about a played disjunction.
	φ_i	**II** makes a choice of $i \in I$.
$\varphi(c)$		**I** tests a played $\forall x \varphi(x)$ by choosing $c \in C$.
	$\varphi(c)$	**II** confirms.
$\exists x \varphi(x)$		**I** enquires about a played existential statement.
	$\varphi(c)$	**II** makes a choice of $c \in C$.
t		**I** enquires about a constant $L \cup C$-term t.
	$\approx ct$	**II** makes a choice of $c \in C$.

Figure 8.1 The game $\mathrm{MEG}(T, L)$.

Definition 8.11 The Model Existence Game $\mathrm{MEG}(\varphi, L)$ for a countable vocabulary L and a sentence φ of $L_{\omega_1 \omega}$ is the game $G_\omega(W)$ where W consists of sequences $(x_0, y_0, x_1, y_1, \ldots)$ where player **II** has followed the rules of Figure 8.1 and for no atomic $L \cup C$-sentence ψ both ψ and $\neg\psi$ are in $\{y_0, y_1, \ldots\}$.

We now extend the first leg of the Strategic Balance of Logic, the equivalence between the Semantic Game and the Model Existence Game, from first-order logic to infinitary logic:

Theorem 8.12 (Model Existence Theorem for $L_{\omega_1\omega}$) *Suppose L is a countable vocabulary and φ is an L-sentence of $L_{\omega_1\omega}$. The following are equivalent:*

(1) There is an L-structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$.
(2) Player **II** *has a winning strategy in* $\mathrm{MEG}(\varphi, L)$.

Proof The implication $(1) \to (2)$ is clear as **II** can keep playing sentences that are true in $\mathcal{M}$. For the other implication we proceed as in the proof of Theorem 6.35. Let $C = \{c_n : n \in \mathbb{N}\}$ and $Trm = \{t_n : n \in \mathbb{N}\}$. Let $(x_0, y_0, x_1, y_1, \ldots)$ be a play in which player **II** has used her winning strategy and player **I** has maintained the following conditions:

1. If $n = 0$, then $x_n = \varphi$.
2. If $n = 2 \cdot 3^i$, then x_n is $\approx c_i c_i$.
3. If $n = 4 \cdot 3^i \cdot 5^j \cdot 7^k \cdot 11^l$, y_i is $\approx c_j t_k$, and y_l is $\varphi(c_j)$, then x_n is $\varphi(c_i)$.
4. If $n = 8 \cdot 3^i \cdot 5^j$ and y_i is $\bigwedge_{m\in\mathbb{N}} \varphi_m$, then x_n is φ_j.
5. If $n = 16 \cdot 3^i$ and y_i is $\bigvee_{m\in\mathbb{N}} \varphi_m$, then x_n is $\bigvee_{m\in\mathbb{N}} \varphi_m$.
6. If $n = 32 \cdot 3^i \cdot 5^j$, y_i is $\forall x\varphi(x)$, then x_n is $\varphi(c_j)$.
7. etc.

The rest of the proof is exactly as in the proof of Theorem 6.35. □

Our success in the above proof is based on the fact that even if we deal with infinitary formulas we can still manage to let player **I** list all possible formulas that are relevant for the consistency of the starting formula. If even one uncountable conjunction popped up, we would be in trouble.

It suffices to consider in $\mathrm{MEG}(\varphi, L)$ such constant terms t that are either constants or contain no other constants than those of C. Moreover, we may assume that if player **I** enquires about $\approx tt$, then $t = c_n$ for some $n \in \mathbb{N}$.

Corollary *Let L be a countable vocabulary. Suppose φ and ψ are sentences of $L_{\omega_1\omega}$. The following are equivalent:*

(1) $\varphi \models \psi$.
(2) Player **I** *has a winning strategy in* $\mathrm{MEG}(\varphi \wedge \neg\psi, L)$.

The proof of the Compactness Theorem does not go through, and should not, because there are obvious counter-examples to compactness in $L_{\omega_1\omega}$. In many proofs where one would like to use the Compactness Theorem one can instead use the Model Existence Theorem. The non-definability of well-order in $L_{\infty\omega}$ was proved already in Theorem 7.26 but we will now prove a stronger version for $L_{\omega_1\omega}$:

Theorem 8.13 (Undefinability of Well-Order) *Suppose L is a countable vocabulary containing a unary predicate symbol U and a binary predicate symbol $<$, and $\varphi \in L_{\omega_1\omega}$. Suppose that for all $\alpha < \omega_1$ there is a model $\mathcal{M}$ of φ such that $(\alpha, <) \subseteq (U^{\mathcal{M}}, <^{\mathcal{M}})$. Then φ has a model $\mathcal{N}$ such that $(\mathbb{Q}, <) \subseteq (U^{\mathcal{N}}, <^{\mathcal{N}})$.*

Proof Let $D = \{d_r : r \in \mathbb{Q}\}$ be a set of new constant symbols. Let us call them d-constants. Let $\theta = \bigwedge_{r<s}(d_r < d_s)$. We show that player **II** has a winning strategy in

$$\mathrm{MEG}(\varphi \wedge \theta, L \cup D).$$

This clearly suffices. The strategy of **II** is the following: Suppose she has played $\{y_0, \ldots, y_{n-1}\}$ so far and $y_i = \theta$ or

$$y_i = \varphi_i(c_0, \ldots, c_m, d_{r_1}, \ldots, d_{r_l}),$$

where $d_{r_1}, \ldots, d_{r_l}$ are the d-constants appearing in $\{y_0, \ldots, y_{n-1}\}$ except in θ. She maintains the following condition:

($\star$) For all $\alpha < \omega_1$ there is a model $\mathcal{M}$ of φ and $b_1, \ldots, b_l \in U^{\mathcal{M}} \subseteq \omega_1$ such that

$$\mathcal{M} \models \exists x_0 \ldots \exists x_m \bigwedge_{i<n} \varphi_i(x_0, \ldots, x_m, b_1, \ldots, b_l)$$

and

$$\alpha \leq b_1, b_1 + \alpha \leq b_2, \ldots, b_{l-1} + \alpha \leq b_l.$$

We show that player **II** can indeed maintain this condition.

For most moves of player **I** the move of **II** is predetermined and we just have to check that ($\star$) remains valid. For a start, if **I** plays φ, condition ($\star$) holds by assumption. If **I** enquires about substitution or plays a conjunct of a played conjunction, no new constants are introduced, so ($\star$) remains true. Also, if **I** tests a played $\forall x\varphi(x)$ or enquires about a played $\exists x\varphi(x)$, no new constants of D are introduced, so ($\star$) remains true. We may assume that **I** enquires about $\approx tt$ only if $t = c_n$ and so ($\star$) holds by the induction hypothesis. Let us then

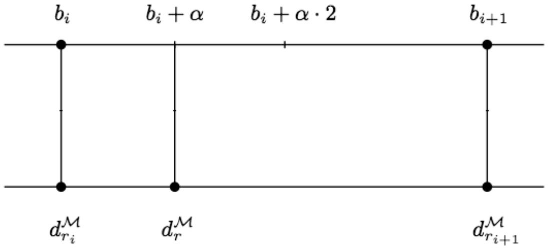


Figure 8.2

assume ($\star$) holds and **I** enquires about a played disjunction $\bigvee_{i\in I}\psi_i$. For each $\alpha < \omega_1$ we have a model $\mathcal{M}_\alpha$ as in ($\star$) and some $i_\alpha \in I$ such that $\mathcal{M}_\alpha \models \psi_{i_\alpha}$. Since **I** is countable, there is a fixed $i \in I$ such that for uncountably many $\alpha < \omega_1$: $\mathcal{M}_\alpha \models \psi_i$. If **II** plays this ψ_i, condition ($\star$) is still true.

The remaining case is that **I** enquires about a constant term t. We may assume $t = d_r$ as otherwise there is nothing to prove. The constants of D occurring so far in the game are $d_{r_1}, \ldots, d_{r_l}$. Let us assume $r_i < r < r_{i+1}$. To prove ($\star$), assume $\alpha < \omega_1$ and let $\beta = \alpha \cdot 2$. By the induction hypothesis there is $\mathcal{M}$ as in ($\star$) such that $b_i + \beta \le b_{i+1}$. Let d_r be interpreted in $\mathcal{M}$ as $b_i + \alpha$. Now $\mathcal{M}$ satisfies the condition ($\star$) (see Figure 8.3).

□

The following corollary is due to Lopez-Escobar (1966b).

Corollary *If φ is a sentence of $L_{\omega_1\omega}$ in a vocabulary which contains the unary predicate U and the binary predicate $<$, and $(U^{\mathcal{M}}, <^{\mathcal{M}})$ is well-ordered in every model of φ, then there is $\alpha < \omega_1$ such that the order type of the structure $(U^{\mathcal{M}}, <^{\mathcal{M}})$ is $< \alpha$ for every model $\mathcal{M}$ of φ.*

Corollary *The class of well-orderings is not a PC-class of $L_{\omega_1\omega}$.*

The undefinability of well-ordering as a PC-class of $L_{\infty\omega}$ will be established later. We now prove the Craig Interpolation Theorem for $L_{\omega_1\omega}$. There are several different proofs of this theorem, some of which employ the above corollary directly. Our proof is like the original proof by Lopez-Escobar, except that we operate with Model Existence Games instead of Gentzen systems.

Theorem 8.14 (Separation Theorem) *Suppose L_1 and L_2 are vocabularies. Suppose φ is an L_1-sentence of $L_{\omega_1\omega}$ and ψ is an L_2-sentence of $L_{\omega_1\omega}$ such* *that* $\models \varphi \to \psi$. *Then there is an* $L_1 \cap L_2$*-sentence* θ *of* $L_{\omega_1\omega}$ *such that* $\models \varphi \to \theta$ *and* $\models \theta \to \psi$.

Proof This is similar to the proof of Theorem 6.40. We assume, w.l.o.g., that L_1 and L_2 are relational. Let $L = L_1 \cap L_2$. We describe, assuming that no such θ exists, a winning strategy of **II** in $\mathrm{MEG}(\varphi \wedge \neg\psi, L_1 \cup L_2)$. We now follow closely the proof of Theorem 6.40, where the strategy of **II** was to divide the set Ψ of her moves into two parts S_1^n and S_2^n such that S_1^n consists of the $L_1 \cup C$-sentences of Ψ and S_2^n consists of the $L_2 \cup C$-sentences of Ψ. In addition it is assumed that

(*) There is no $L \cup C$-sentence θ that separates S_1^n and S_2^n.

There are two new cases over and above those of Theorem 6.40:
Case 5′. Player **I** plays φ_i where for example $\bigwedge_{i\in I} \varphi_i \in S_1^n$. Let $S_1^{n+1} = S_1^n \cup \{\varphi_i\}$ and $S_2^{n+1} = S_2^n$. If θ separates S_1^{n+1} and S_2^{n+1}, then clearly θ also separates S_1^n and S_2^n.
Case 6′. Player **I** plays $\bigvee_{i\in I} \varphi_i$, where for example $\bigvee_{i\in I} \varphi_i \in S_1^n$. We claim that for some $i \in I$ the sets $S_1^n \cup \{\varphi_i\}$ and S_2^n satisfy (*). Otherwise there is for each $i \in I$ some θ_i that separates $S_1^n \cup \{\varphi_i\}$ and S_2^n. Let $\theta = \bigvee_{i\in I} \theta_i$. Then θ separates S_1^n and S_2^n contrary to assumption.

□

8.4 Large Models

Suppose Γ is an L-fragment of $L_{\kappa^+\omega}$ of size κ and $\mathcal{M}$ is an L-structure of size $> \kappa$. If we define on M

$$a \sim b \iff \text{for every } \varphi(x) \in \Gamma : \mathcal{M} \models \varphi(a) \iff \mathcal{M} \models \varphi(b)$$

then by the Pigeonhole Principle there is a subset I of M of size $> \kappa$ such that for $a, b \in I$ and $\varphi(x) \in \Gamma$:

$$\mathcal{M} \models \varphi(a) \iff \mathcal{M} \models \varphi(b).$$

We say that the set I is *indiscernible* in $\mathcal{M}$ with respect to Γ. If we want indiscernibility relative to formulas with more than one free variable, we have to use *Ramsey theory*.

Definition 8.15 Suppose L is a vocabulary, $\mathcal{M}$ is an L-structure, and Γ is an L-fragment. A linear order $(I, <)$, where $I \subseteq M$, is Γ-*indiscernible* in $\mathcal{M}$ if for all $a_1 < \ldots < a_n, b_1 < \ldots < b_n$ in I and any $\varphi(x_1, \ldots, x_n)$ in Γ:

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n) \iff \mathcal{M} \models \varphi(b_1, \ldots, b_n).$$

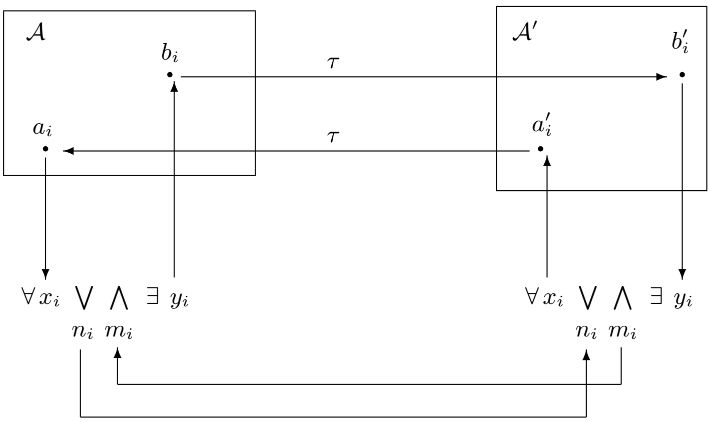


Figure 8.9 Strategy ρ.

Although elementary equivalence relative to $L_{\infty G}$ coincides with elementary equivalence relative to $L_{\infty\omega}$, there are properties expressible in $L_{\infty G}$ which are not expressible even in $L_{\infty\infty}$, as we shall prove in Proposition 9.38.

8.7 Historical Remarks and References

Good sources for the model theory of infinitary logic are Keisler (1971) and Makkai (1977). Theorem 8.10 is from Kueker (1972, 1977). Theorem 8.12 is from Keisler (1971), where the method of consistency properties is first presented in the context of infinitary logic. Subsequently it was extensively used in infinitary logic in Makkai (1969b,a), Harnik and Makkai (1976), and Green (1975). Theorem 8.13 and its two vorollaries are from Lopez-Escobar (1966b,a). The strong formulation of Theorem 8.13 is from Keisler (1971). Theorem 8.14 is from Lopez-Escobar (1965).

By means of the Model Existence Game (or consistency properties) many variations of the Craig Interpolation Theorem can be proved for $L_{\omega_1\omega}$, as demonstrated convincingly in Keisler (1971). We have collected some of them in Exercises 8.10–8.17.

Theorem 8.22 is from Morley (1968). Theorem 8.31 goes back to Morley (1968), but the present proof is due to Shelah. Proposition 8.32 is from Lopez-Escobar (1966b,a). Proposition 8.41 is essentially due to Keisler (1965). A useful source for game quantifiers is Burgess (1977). Approximations of game formulas were first considered in set theory in Kuratowski (1966). For more on Souslin–formulas, see Burgess (1978) and Green (1978).

Exercises 8.10–8.12 and 8.16 are from Lopez-Escobar (1965). Exercises 8.13-8.14 are from Malitz (1969). Exercise 8.15 is from Barwise (1969). Exercise 8.17 is from Makkai (1969b), which is a good source for all the Exercises 8.10–8.17. Exercise 8.30 is from Sierpiński (1933). Exercise 8.46 is from Burgess (1978), where also related results are proved. Exercise 8.47 is from Green (1979).

Exercises

8.1 Find an uncountable model $\mathcal{M}$ such that there is no countable $\mathcal{N}$ with $\mathcal{N} \equiv_{\infty\omega} \mathcal{M}$.

8.2 Let L be the vocabulary $\{E\} \cup \{c_n : n \in \mathbb{N}\}$, where each c_n is a constant symbol and E is a binary relation symbol. Let for $A \subseteq \mathbb{N}$ the sentence φ_A be

$$\left(\bigwedge_{n<m<\omega} \neg{\approx} c_n c_m\right) \wedge \exists x_0 \forall x_1 \left(x_1 E x_0 \leftrightarrow \bigvee_{n \in A} {\approx} c_n x_1\right).$$

Let $\Phi = \{\varphi_A : A \subseteq \mathbb{N}\}$. Show that the sentence $\bigwedge\{\varphi : \varphi \in \Phi \cap X\}$ has a model whatever X is, but it has a countable model if and only if X is countable.

8.3 Suppose $\mathcal{M} = (\alpha + \alpha, <)$. Show that $\mathcal{M}^X \cong (\beta_X + \beta_X, <)$ for some β_X for almost all X.

8.4 Prove the claim of Example 8.4.

8.5 A class K of models is *closed* if $\mathcal{M} \in K$ if and only if $\mathcal{M}^X \in K$ for almost all X. Suppose a closed class contains, up to isomorphism, only countably many countable models. Show that it is definable in $L_{\omega_1\omega}$.

8.6 Show that the class of models $(\mathcal{M}, P)$, where P is $L_{\infty\omega}$-definable on $\mathcal{M}$, is a closed class.

8.7 Show that the class of countable well-orders is the union of two disjoint closed classes.

8.8 Suppose $(A, <)$ is an uncountable linear order. Show that $(A, <)$ contains a copy of ω_1, a copy of ω_1^*, or a copy of the rationals. (Hint: Assume not. Prove first that there is $a \in A$ such that both $(\leftarrow, a]$ and $[a, \rightarrow)$ are uncountable in $(A, <)$.)

8.9 Show that if φ is a sentence of $L_{\omega_1\omega}$ in a vocabulary which contains the unary predicate U and the binary predicate $<$ and φ has a model $\mathcal{M}$ with

$(U^{\mathcal{M}}, <^{\mathcal{M}})$ an uncountable linear order, then φ has a model $\mathcal{N}$ such that $(U^{\mathcal{N}}, <^{\mathcal{N}})$ contains a copy of the rationals.

8.10 (Lyndon Interpolation Theorem) Suppose φ and ψ are sentences in $L_{\omega_1\omega}$ and $\models \varphi \to \psi$. Show that there is θ in $L_{\omega_1\omega}$ such that $\models \varphi \to \theta$, $\models \theta \to \psi$, every relation symbol occurring positively (negatively) in θ occurs positively (negatively) in φ and ψ.

8.11 Assume in Exercise 8.10 the sentences φ and ψ have no function or constant symbols, and no identity. Assume also $\not\models \neg\varphi$ and $\not\models \psi$. Show that θ can be chosen so that it does not contain identity.

8.12 Suppose φ and ψ are sentences of $L_{\omega_1\omega}$ such that if $\mathcal{M}$ and $\mathcal{N}$ are models of φ, $\mathcal{N}$ is a homomorphic image of $\mathcal{M}$, and $\mathcal{M} \models \psi$, then $\mathcal{N} \models \psi$. Show that there is a positive $L_{\omega_1\omega}$- sentence θ such that $\varphi \models \psi \leftrightarrow \theta$.

8.13 Suppose L_1 and L_2 are vocabularies which contain no function symbols. Let φ be an L_1-sentence and ψ an L_2-sentence of $L_{\omega_1\omega}$ such that ψ is universal and $\models \varphi \to \psi$. Show that there is a universal $L_1 \cap L_2$-sentence θ of $L_{\omega_1\omega}$ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$.

8.14 Suppose φ and ψ are sentences of $L_{\omega_1\omega}$ such that if $\mathcal{M}$ and $\mathcal{N}$ are models of φ, $\mathcal{N}$ is a submodel of $\mathcal{M}$, and $\mathcal{M} \models \psi$, then $\mathcal{N} \models \psi$. Show that there is a universal sentence θ of $L_{\omega_1\omega}$ such that $\varphi \models \psi \leftrightarrow \theta$.

8.15 Suppose φ and ψ are sentences of $L_{\omega_1\omega}$. Show that every countable model of φ can be embedded in some countable model of ψ if and only if every universal logical consequence of ψ in $L_{\omega_1\omega}$ is a logical consequence of φ.

8.16 Suppose φ and ψ are sentences of $L_{\omega_1\omega}$. Show that every homomorphic image of a model of φ is a model of ψ if and only if there is a positive sentence θ in $L_{\omega_1\omega}$ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$.

8.17 The class K of countable structures $\mathcal{B}$ such that $\mathcal{B}$ is isomorphic to a substructure of some model $\mathcal{A}$ of $\varphi \in L_{\omega_1\omega}$ is identical to the class of all countable models of the set of universal sentences $\theta \in L_{\omega_1\omega}$ such that $\varphi \models \psi$.

8.18 Consider the following game $G^{\kappa}_{\mathrm{cub}}(\mathcal{C})$ where $\mathcal{C}$ is a set of subsets of M of cardinality $\leq \kappa$. Players **I** and **II** play as in $G_{\mathrm{cub}}(\mathcal{C})$ but the game goes on for κ rounds producing a set $X = \{x_\alpha : \alpha < \kappa\} \cup \{y_\alpha : \alpha < \kappa\}$. Player **II** wins if $X \in \mathcal{C}$. Show that if **II** has a winning strategy in $G^{\kappa}_{\mathrm{cub}}(\mathcal{C})$ and $G^{\kappa}_{\mathrm{cub}}(\mathcal{D})$, then she has a winning strategy in $G^{\kappa}_{\mathrm{cub}}(\mathcal{C} \cap \mathcal{D})$.

8.19 Show that if $\mathcal{F}$ is a set of finitary functions on M, $|\mathcal{F}| = \kappa$, and $\mathcal{C}$ is the set of $X \subseteq M$ of cardinality κ closed under each function in $\mathcal{F}$, then **II** has a winning strategy in $G^{\kappa}_{\mathrm{cub}}(\mathcal{C})$. Use this to conclude that if $\mathcal{M}$ is an L-structure, $|L| \leq \kappa$, $\varphi \in L_{\kappa^+\omega}$, $\mathcal{M} \models \varphi$ and $\mathcal{C}$ is the set of domains of $\mathcal{M}_0 \subseteq \mathcal{M}$ with $\mathcal{M}_0 \models \varphi$, then **II** has a winning strategy in $G^{\kappa}_{\mathrm{cub}}(\mathcal{C})$.

8.20 Show that we can extend each vocabulary L to L^*, translate each $L_{\kappa^+\omega}$-sentence φ in the vocabulary L to an $L_{\kappa^+\omega}$-sentence φ^* in the vocabulary L^*, expand each L-structure $\mathcal{M}$ to an L^*-structure $\mathcal{M}^*$, and extend any L-fragment $\mathcal{T}$ to a set $\mathcal{T}^*$ of cardinality κ of $L_{\kappa^+\omega}$-sentences in the vocabulary L^* such that for all L-structures $\mathcal{M}$, L^*- structures $\mathcal{N}$, L-fragments $\mathcal{T}$ and $L_{\kappa^+\omega}$-sentences $\varphi \in \mathcal{T}$ in the vocabulary L:

(1) $\mathcal{M} \models \varphi \quad \Longrightarrow \quad \mathcal{M}^* \models \mathcal{T}^* \cup \{\varphi^*\}$.
(2) $\mathcal{N} \models \mathcal{T}^* \cup \{\varphi^*\} \quad \Longrightarrow \quad \mathcal{N} \restriction L \models \varphi$.
(3) φ^* is quantifier-free.
(4) $\mathcal{T}^*$ is a set of universal sentences.

8.21 Suppose L is a vocabulary of cardinality $\leq \kappa$, φ is a sentence of $L_{\kappa^+\omega}$ in the vocabulary L, $\mathcal{M}$ is an L-structure such that $\mathcal{M} \models \varphi$, and $\kappa \leq \mu \leq |M|$. Show that there is $\mathcal{M}_0 \subseteq \mathcal{M}$ such that $\mathcal{M}_0 \models \varphi$ and $|M_0| = \mu$.

8.22 Find a counter-example to the following claim: Suppose L is a vocabulary of cardinality $\leq \kappa$, $\alpha < \kappa^+$, and $\mathcal{M}$ is an L-structure. Then there is $\mathcal{M}_0 \subseteq \mathcal{M}$ such that $\mathcal{M}_0 \simeq_p^\alpha \mathcal{M}$ and $|M_0| \leq \kappa$.

8.23 Prove that the class of models (M, A, B), where $A, B \subseteq M$ and $|A| < |B|$ is not RPC in $L_{\infty\omega}$. In fact, prove the following statement: If $\varphi \in L_{\kappa^+\omega}$ has a model $\mathcal{M}$ such that $\kappa \leq |U^{\mathcal{M}}| < |V^{\mathcal{M}}|$, then φ has a model $\mathcal{N}$ such that $|U^{\mathcal{N}}| = |V^{\mathcal{N}}|$.

8.24 Show that the class of linear orders with uncountable cofinality is not PC-definable in $L_{\infty\omega}$. In fact, prove the following statement: If $\varphi \in L_{\kappa^+\omega}$ has a model $\mathcal{M}$ such that $(U^{\mathcal{M}}, <^{\mathcal{M}})$ has cofinality κ^+, then φ has a model $\mathcal{N}$ such that $(U^{\mathcal{N}}, <^{\mathcal{N}})$ has cofinality $\aleph_0$. (A linear order $(M, <)$ has *cofinality* κ (or is κ-cofinal) if κ is the smallest cardinal for which there is $A \subseteq M$ such that $|A| = \kappa$ and A has no upper bound in $(M, <)$.)

8.25 Prove that the class of graphs (G, E) which have no countable cover (i.e. there is no countable subset C such that for every $a \in G$ there is some $y \in C$ with xEy) is not RPC in $L_{\infty\omega}$. In fact, prove the following statement: If $\varphi \in L_{\kappa^+\omega}$ has, for every graph (G, E) of cardinality κ^+ without a countable cover, a model $\mathcal{M}$ such that $(U^{\mathcal{M}}, R^{\mathcal{M}}) \cong (G, E)$, then φ has a model $\mathcal{N}$ such that $(U^{\mathcal{M}}, R^{\mathcal{M}})$ has a countable cover.

8.26 Show that there is a sentence in $L_{\kappa^+\omega}$ which has a model of size κ^+ but none of size $> \kappa^+$.

8.27 Show that there is a sentence in $L_{\kappa^+\omega}$ which has a model of size κ^{++} but none of size $> \kappa^{++}$.

8.28 Show that there is a sentence in $L_{\kappa^+\omega}$ which has a model of size 2^κ but none of size $> 2^\kappa$.

8.29 Show that there is a sentence in $L_{\kappa^+\omega}$ which has a model of size 2^{2^κ} but none of size $> 2^{2^\kappa}$.

8.30 Show $2^\omega \not\to (\omega_1)^2_2$. (Hint: Take a well-ordering $\lhd$ of $\mathbb{R}$. Then define a coloring of pairs $\{x, y\}$ of reals by reference to $\lhd$ and the standard ordering of $\mathbb{R}$.)

8.31 Show that if $\kappa \to (\kappa)^2_2$, then κ is regular.

8.32 Prove directly $6 \to (3)^2_2$.

8.33 Show $15 \not\to (4)^2_2$ (it is probably just as easy to show $17 \not\to (4)^2_2$.)

8.34 Deduce the undefinability of well-order in $L_{\omega_1\omega}$ directly from Theorem 8.22. (Hint: Assume well-order could be defined in $L_{\omega_1\omega}$. Show that then there is a sentence in $L_{\omega_1\omega}$ with a model of size $\beth_{\omega_1}$ but none bigger.)

8.35 Show that for every vocabulary L, every L-fragment Γ of $L_{\infty\omega}$ and every L-structure $\mathcal{M}$ there is $L' \supseteq L$, an L'-fragment Γ' such that $\Gamma' \supseteq \Gamma$, and an expansion $\mathcal{M}'$ of $\mathcal{M}$ to an L'-structure such that $|L'| = |L| + \aleph_0$, $|\Gamma'| = |\Gamma| + \aleph_0$ and $\mathcal{M}'$ has Skolem functions for all $\varphi \in \Gamma'$.

8.36 Construct for each infinite κ a linear order with 2^κ automorphisms.

8.37 An element a of a Boolean algebra $\mathcal{M}$ is an *atom* if for all $b \in M : 0 \leq b \leq a$ implies $0 = b$ or $b = a$. Show that if $\mathcal{M}$ is a Boolean algebra and $(I, <)$ is a linear order such that I is a set of atoms of $\mathcal{M}$, then $(I, <)$ is Γ-indiscernible in $\mathcal{M}$ for any fragment Γ of $L_{\infty\omega}$.

8.38 Suppose $\mathcal{M}$ is an equivalence relation and $(I, <)$ is a linear order such that I is included in one of the equivalence classes of $\mathcal{M}$. Show that $(I, <)$ is Γ-indiscernible in $\mathcal{M}$ for all fragments Γ of $L_{\infty\omega}$. Is the same true if I is a set of non-$\mathcal{M}$-equivalent elements of M?

8.39 Suppose L is a vocabulary, $\mathcal{M}$ and $\mathcal{M}'$ L-structures, Γ an L-fragment of $L_{\infty\omega}$, $(I, <_I)$ Γ-indiscernible in $\mathcal{M}$, $(J, <_J)$ Γ-indiscernible in $\mathcal{N}$, and $\mathcal{M} \models \varphi(a_1, \ldots, a_n) \iff \mathcal{N} \models \varphi(b_1, \ldots, b_n)$ for all atomic φ, $a_1 <_I \cdots <_I a_n$ and $b_1 <_J \cdots <_J b_n$. Suppose $(I, <_I) \simeq_p (J, <_J)$. Show that $[I]_\mathcal{M} \simeq_p [J]_\mathcal{N}$.

8.40 Suppose $\varphi \in L_{\omega_1\omega}$ has for each $\alpha < \omega_1$ a model of size $\beth_\alpha$. Show that φ has arbitrarily large models all of which are partially isomorphic. (Hint: Use the previous exercise.)

8.41 Show that the Scott rank of $(\alpha, <)$ is always α.

8.42 Show that the Scott rank of $(\mathbb{Q}, <)$ is ω.

8.43 Show that the two truth definitions of game logic are equivalent.

8.44 Show that the conjunction and disjunction of two formulas of the form (8.18) is again of the form (8.18), up to logical equivalence.

8.45 Use Theorem 8.47 to give a quick proof of the Craig Interpolation Theorem.

Proof Suppose T is a bottleneck. Let $\alpha < \kappa^+$ such that $T \in V[G_\alpha]$. Let A_α be the Cohen subset of κ added at stage α. Note that A_α is a bistationary subset of κ. We first show that $\Vdash T \not\leq T(A_\alpha)$. Suppose

$$p \Vdash \hat{f} : T(A_\alpha) \to T \text{ is strictly increasing.}$$

When we force with A_α, calling the forcing notion $\mathcal{P}'$, an uncountable branch appears in $T(A_\alpha)$, hence also in T. The product forcing $\mathcal{P}_\alpha \star \mathcal{P}'$ contains a κ-closed dense set (Exercise 9.45). Hence it cannot add a branch of length κ to T. We have shown that $T(A_\alpha) \not\leq T$ in $V[G]$. Since T is a bottleneck, $T \leq T(A_\alpha)$. By repeating the same with $-A_\alpha$ we get $T \leq T(-A_\alpha)$. In sum, $T \leq T(A_\alpha) \otimes T(-A_\alpha)$ (see Exercise 9.44 for the definition of $\otimes$). But $T(A_\alpha) \otimes T(-A_\alpha) \leq T_p^\kappa$ (Exercise 9.46). Hence $T \leq T_p^\kappa$. □

It is also known (Todorčević and Väänänen (1999)) that if $V = L$, then there are no bottlenecks in the class $\mathcal{T}_{\aleph_1,\aleph_1}$ above $T_p^{\aleph_1}$.

9.5 The Transfinite Dynamic Ehrenfeucht–Fraïssé Game

In this section we introduce a more general form of the Ehrenfeucht–Fraïssé Game. The new game generalizes both the usual Ehrenfeucht–Fraïssé Game and the dynamic version of it. In this game player **I** makes moves not only in the models in question but also moves up a po-set, move by move. The game goes on as long as **I** can move. This game generalizes at the same time the games $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ and $\mathrm{EFD}_\delta(\mathcal{A}_0, \mathcal{A}_1)$. Therefore we denote it by $\mathrm{EF}_\mathcal{P}$ rather than by $\mathrm{EFD}_\mathcal{P}$.

If $\mathcal{P}$ is a po-set, let $\mathrm{b}(\mathcal{P})$ denote the least ordinal δ so that $\mathcal{P}$ does not have an ascending chain of length δ.

Definition 9.66 Suppose $\mathcal{A}_0$ and $\mathcal{A}_1$ are L-structures and $\mathcal{P}$ is a po-set. The *Transfinite Dynamic Ehrenfeucht–Fraïssé Game* $\mathrm{EF}_\mathcal{P}(\mathcal{A}_0, \mathcal{A}_1)$ is like the game $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$ except that on each round **I** chooses an element $c_\alpha \in \{0, 1\}$, an element $x_\alpha \in A_{c_\alpha}$, and an element $p_\alpha \in \mathcal{P}$. It is required that

$$p_0 <_\mathcal{P} \ldots <_\mathcal{P} p_\alpha <_\mathcal{P} \ldots.$$

Finally **I** cannot play a new p_α anymore because $\mathcal{P}$ is a set. Suppose **I** has played $\bar{z} = \langle (c_\beta, x_\beta) : \beta < \alpha \rangle$ and **II** has played $\bar{y} = \langle y_\beta : \beta < \alpha \rangle$. If $p_{\bar{z},\bar{y}}$ is a partial isomorphism between $\mathcal{A}_0$ and $\mathcal{A}_1$, **II** has won the game, otherwise **I** has won.

Thus a winning strategy of **I** in $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ is a sequence $\rho = \langle \rho_\alpha : \alpha < \mathrm{b}(\mathcal{P})\rangle$ and a strategy of **II** is a sequence $\tau = \langle \tau_\alpha : \alpha < \mathrm{b}(\mathcal{P})\rangle$. Note that

$$\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1) \text{ is the same game as } \mathrm{EF}_{(\alpha,<)}(\mathcal{A}_0, \mathcal{A}_1),$$

and

$$\mathrm{EFD}_\alpha(\mathcal{A}_0, \mathcal{A}_1) \text{ is the same game as } \mathrm{EF}_{(\alpha,>)}(\mathcal{A}_0, \mathcal{A}_1).$$

Naturally, if α is finite, the games $\mathrm{EF}_{(\alpha,<)}(\mathcal{A}_0, \mathcal{A}_1)$ and $\mathrm{EF}_{(\alpha,>)}(\mathcal{A}_0, \mathcal{A}_1)$ are one and the same game. But if α happens to be infinite, there is a big difference: The first is a transfinite game while the second can only go on for a finite number of moves.

The ordering $\mathcal{P} \le \mathcal{P}'$ of po-sets has a close connection to the question who wins the game $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$, as the following two results manifest:

Lemma 9.67 *If* **II** *wins the game* $\mathrm{EF}_{\mathcal{P}'}(\mathcal{A}_0, \mathcal{A}_1)$ *and* $\mathcal{P} \le \mathcal{P}'$, *then* **II** *wins the game* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$. *If* **I** *wins the game* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ *and* $\mathcal{P} \le \mathcal{P}'$, *then* **I** *wins the game* $\mathrm{EF}_{\mathcal{P}'}(\mathcal{A}_0, \mathcal{A}_1)$.

Proof Exercise 9.50. □

Proposition 9.68 *Suppose* **II** *wins* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ *and* **I** *wins* $\mathrm{EF}_{\mathcal{P}'}(\mathcal{A}_0, \mathcal{A}_1)$. *Then* $\sigma\mathcal{P} \le \mathcal{P}'$.

Proof Suppose **II** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ with τ and **I** wins $\mathrm{EF}_{\mathcal{P}'}(\mathcal{A}_0, \mathcal{A}_1)$ with ρ. We describe a winning strategy of **I** in $G(\mathcal{P}', \mathcal{P})$, and then the claim follows from Lemma 9.60. Suppose $\rho_0(\emptyset) = (c_0, x_0, p'_0)$. The element p'_0 is the first move of **I** in $G(\mathcal{P}', \mathcal{P})$. Suppose **II** plays $p_0 \in \mathcal{P}$. Let

$$\begin{aligned} y_0 &= \tau_0(((c_0, x_0, p_0))), \\ (c_1, x_1, p'_1) &= \rho_1((y_0)). \end{aligned}$$

The element p'_1 is the second move of **I** in $G(\mathcal{P}', \mathcal{P})$. More generally the equations

$$\begin{aligned} y_\beta &= \tau_\beta(\langle (c_\gamma, x_\gamma, p_\gamma) : \gamma \le \beta\rangle) \\ (c_\alpha, x_\alpha, p'_\alpha) &= \rho_\alpha(\langle y_\beta : \beta < \alpha\rangle) \end{aligned}$$

define the move p'_α of **I** in $G(\mathcal{P}', \mathcal{P})$ after **II** has played $\langle p_\beta : \beta < \alpha\rangle$. The game can only end if **II** cannot move p_α at some point, so **I** wins. □

Suppose $\mathcal{A}_0 \not\cong \mathcal{A}_1$. Then there is a least ordinal

$$\delta \le \mathrm{Card}(A_0) + \mathrm{Card}(A_1)$$

such that **II** does not win $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$. Thus for all $\alpha + 1 < \delta$ there is a winning strategy for **II** in $\mathrm{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$. Let $K(= K(\mathcal{A}_0, \mathcal{A}_1))$ be the set of all winning strategies of **II** in $\mathrm{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$ for $\alpha + 1 < \delta$. We can make K a tree by letting

$$\langle \tau_\xi : \xi \le \alpha \rangle \le \langle \tau'_\xi : \xi \le \alpha' \rangle$$

if and only if $\alpha \le \alpha'$ and $\forall \xi \le \alpha(\tau_\xi = \tau'_\xi)$.

Definition 9.69 We call K, as defined above, the *canonical Karp tree* of the pair $(\mathcal{A}_0, \mathcal{A}_1)$.

Note that even when δ is a limit ordinal K does not have a branch of length δ, for otherwise **II** would win $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$.

Lemma 9.70 *Suppose $\mathcal{P}$ is a po-set. Then*

$$\exists \textit{ wins } \mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1) \iff \sigma'\mathcal{P} \le K.$$

Proof $\Rightarrow$ Suppose **II** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ with τ. If $s = \langle s_\xi : \xi \le \alpha \rangle \in \sigma'\mathcal{P}$, we can define a strategy τ' of **II** in $\mathrm{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$ as follows

$$\tau'_\xi(\langle (c_\eta, x_\eta) : \eta \le \xi \rangle) = \tau_\xi(\langle (c_\eta, x_\eta, s_\eta) : \eta \le \xi \rangle).$$

Since K does not have a branch of length δ, $\alpha < \delta$, and hence $\tau' \in K$. The mapping $s \mapsto \tau'$ is an order-preserving mapping $\sigma'\mathcal{P} \to K$.

$\Leftarrow$ Suppose $f : \sigma'\mathcal{P} \to K$ is order-preserving. We can define a winning strategy of **II** in $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ by the equation

$$\tau_\alpha(\langle (c_\xi, x_\xi, s_\xi) : \eta \le \xi \rangle) = f(\langle s_\xi : \xi \le \alpha \rangle)(\langle (c_\xi, x_\xi, s_\eta) : \xi \le \alpha \rangle).$$

□

Proposition 9.71 *Suppose δ is a limit ordinal and* **II** *wins* $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ *for all $\alpha < \delta$. The following are equivalent:*

(i) **II** *wins* $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$.
(ii) **II** *wins* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ *for every po-set $\mathcal{P}$ with no branches of length δ.*

Proof To prove (ii)→(i), suppose **II** does not win $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$. Let $\mathcal{P} = K(\mathcal{A}_0, \mathcal{A}_1)$. Then $\sigma\mathcal{P}$ does not have branches of length δ, hence by (ii) **II** wins $\mathrm{EF}_{\sigma\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ and we get $\sigma\mathcal{P} \le \mathcal{P}$ from Lemma 9.70, a contradiction with Lemma 9.55. The other direction (i)→(ii) is trivial. □

Note Suppose $\kappa = \mathrm{Card}(A_0) + \mathrm{Card}(A_1)$. Then we can compute $\mathrm{Card}(K) \le \sup_{\alpha<\delta}(\kappa^{\kappa^\alpha})^\alpha = \sup_{\alpha<\delta} \kappa^{\kappa^\alpha}$. If GCH and κ is regular, then $\mathrm{Card}(K) \le \kappa^+$. Furthermore, if we assume GCH, we can assume $\mathrm{Card}(\mathcal{P}) \le \kappa$ in (ii) above (Hyttinen). For $\delta = \omega$ this does not depend on GCH. **II** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ if and only if **II** wins $\mathrm{EF}_{\sigma'\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$. So from the point of view of the existence of a winning strategy for **II** we could always assume that $\mathcal{P}$ is a tree.

Corollary **II** *never wins* $\mathrm{EF}_{\sigma K}(\mathcal{A}_0, \mathcal{A}_1)$.

Definition 9.72 A po-set $\mathcal{P}$ is a *Karp po-set* of the pair $(\mathcal{A}_0, \mathcal{A}_1)$ if **II** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ but not $\mathrm{EF}_{\sigma\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$. If a Karp po-set is a tree, we call it a *Karp tree*.

By Lemma 9.70 and the above corollary, there are always Karp trees for every pair of non-isomorphic structures.

Suppose **I** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ with the strategy ρ. Let S_ρ be the set of sequences $\bar{y} = \langle y_\xi : \xi \leq \alpha \rangle \in \mathrm{dom}(\rho)$ such that

$$p_{\rho \restriction \alpha+1, y} \in \mathrm{Part}(\mathcal{A}_0, \mathcal{A}_1).$$

Thus S_ρ is the set of sequences of moves of **II** before she loses $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$, when **I** plays ρ. We can make S_ρ a tree by ordering it as follows

$$\langle y_\xi : \xi \leq \alpha \rangle \leq \langle y'_\xi : \xi \leq \alpha' \rangle$$

if and only if $\alpha \leq \alpha'$ and $\forall \xi \leq \alpha (y_\xi = y'_\xi)$.

Lemma 9.73 **I** *wins* $\mathrm{EF}_{\sigma S_\rho}(\mathcal{A}_0, \mathcal{A}_1)$.

Proof The following equation defines a winning strategy ρ' of **I** in the game $\mathrm{EF}_{\sigma S_\rho}(\mathcal{A}_0, \mathcal{A}_1)$:

$$\rho'_\alpha \langle y_\xi : \xi < \alpha \rangle) = \langle c_\alpha, x_\alpha, \langle (y_\xi : \xi \leq \beta \rangle : \beta < \alpha \rangle,$$

where

$$\rho_\alpha(\langle y_\xi : \xi < \alpha \rangle) = (c_\alpha, x_\alpha, p_\alpha).$$

□

Lemma 9.74 $\sigma S_\rho \leq \mathcal{P}$.

Proof Suppose $s = \langle \langle y_\xi : \xi \leq \beta \rangle : \beta < \alpha \rangle \in \sigma S_\rho$, where

$$\beta_0 < \beta_1 < \ldots < \beta_\eta < \ldots (\eta < \alpha).$$

Let $\delta = \sup_{\eta < \alpha} \beta_\eta$ and

$$\rho_\delta(\langle y_\xi : \xi < \delta \rangle) = (c_\delta, x_\delta, p_\delta).$$

We define $f(s) = p_\delta$. Then $f : \sigma S_\rho \to \mathcal{P}$ is order-preserving. □

Note that Lemma 9.74 implies $\mathcal{P} \not\leq S_\rho$. In particular, if **I** wins $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$ with ρ, then S_ρ is a tree with no branches of length δ.

Suppose $\mathcal{P}_0$ is such that $\sigma\mathcal{P}_0 \leq \mathcal{P}$ and **I** wins $\mathrm{EF}_{\sigma\mathcal{P}_0}$. So $\mathcal{P}_0$ could be S_ρ. Suppose furthermore that there is no $\mathcal{P}_1$ such that $\sigma\mathcal{P}_1 \leq \mathcal{P}_0$ and **I** wins $\mathrm{EF}_{\sigma\mathcal{P}_1}$. Lemma 9.57 implies that this assumption can always be satisfied.

Lemma 9.75 **I** *does not win* $\mathrm{EF}_{\mathcal{P}_0}(\mathcal{A}_0, \mathcal{A}_1)$.

Proof Suppose **I** wins $\mathrm{EF}_{\mathcal{P}_0}(\mathcal{A}_0, \mathcal{A}_1)$ with ρ'. Then **I** wins $\mathrm{EF}_{\sigma S_{\rho'}}(\mathcal{A}_0, \mathcal{A}_1)$ and $\sigma S_{\rho'} \leq \mathcal{P}_0$, contrary to the choice of $\mathcal{P}_0$. □

Definition 9.76 A po-set $\mathcal{P}$ is a *Scott po-set* of $(\mathcal{A}_0, \mathcal{A}_1)$ if **I** wins the game $\mathrm{EF}_{\sigma\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ but not the game $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$. If a Scott po-set is a tree, we call is a *Scott tree*. If $\mathcal{P}$ is both a Scott and a Karp po-set, it is called a *determined* Scott po-set.

By Lemma 9.73 and Lemma 9.75, S_ρ is always a Scott tree of $(\mathcal{A}_0, \mathcal{A}_1)$, so Scott trees always exist. Note that

$$\mathrm{Card}(S_\rho) \leq \sup_{\alpha < \mathrm{b}(\mathcal{P})} (\mathrm{Card}(A_0) + \mathrm{Card}(A_1))^\alpha.$$

Lemma 9.77 *Suppose* **I** *wins* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ *with* ρ *and* $K = K(\mathcal{A}_0, \mathcal{A}_1)$. *Then* $K \leq S_\rho$.

Proof Suppose $\tau \in K$. Let **II** play τ against ρ in $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$. The resulting sequence $\bar{y}$ of moves of **II** is an element of S_ρ. The mapping $\tau \mapsto \bar{y}$ is order-preserving. □

Suppose **II** wins $\mathrm{EF}_{\mathcal{P}_0}(\mathcal{A}_0, \mathcal{A}_1)$ and **I** wins $\mathrm{EF}_{\mathcal{P}_1}(\mathcal{A}_0, \mathcal{A}_1)$ with ρ. Figure 9.7 shows the resulting picture.

In summary, we have proved:

Theorem 9.78 *Suppose* **II** *wins* $\mathrm{EF}_{\mathcal{P}_0}(\mathcal{A}_0, \mathcal{A}_1)$ *and* **I** *wins* $\mathrm{EF}_{\mathcal{P}_1}(\mathcal{A}_0, \mathcal{A}_1)$. *Then there are trees* T_0 *and* T_1 *such that*

(i) $\sigma'\mathcal{P}_0 \leq T_0 \leq T_1 \leq \mathcal{P}_1$.
(ii) **II** *wins* $\mathrm{EF}_{T_0}(\mathcal{A}_0, \mathcal{A}_1)$ *but not* $\mathrm{EF}_{\sigma T_0}(\mathcal{A}_0, \mathcal{A}_1)$.
(iii) **I** *wins* $\mathrm{EF}_{\sigma T_1}(\mathcal{A}_0, \mathcal{A}_1)$ *but not* $\mathrm{EF}_{T_1}(\mathcal{A}_0, \mathcal{A}_1)$.

Example 9.79 Suppose **I** wins $\mathrm{EF}_\omega(\mathcal{A}_0, \mathcal{A}_1)$. By Proposition 7.19 there is a unique $\delta = \delta(\mathcal{A}_0, \mathcal{A}_1)$ such that **II** wins $\mathrm{EF}_{(\delta,>)}(\mathcal{A}_0, \mathcal{A}_1)$ and **I** wins $\mathrm{EF}_{(\delta+1,>)}(\mathcal{A}_0, \mathcal{A}_1)$. Then $(\delta, >)$ is both a Karp and a Scott po-set for $\mathcal{A}_0$ and $\mathcal{A}_1$.

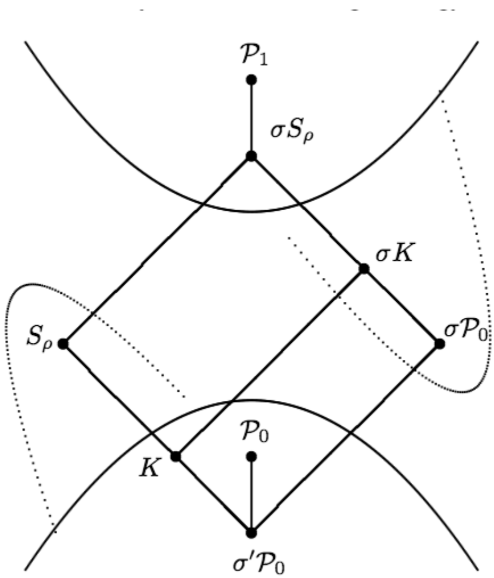


Figure 9.7 The boundary between **II** winning and **I** winning.

Example 9.80 Suppose **II** wins $\text{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ but not $\text{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$. Then $(\alpha, <)$ is a Karp tree (in fact a Karp well-order) of $\mathcal{A}_0$ and $\mathcal{A}_1$. This follows from the fact that $\sigma(\alpha, <) \equiv (\alpha + 1, <)$.

Example 9.81 Suppose **I** wins $\text{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$ but not $\text{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$. Then $(\alpha, <)$ is a Scott tree (in fact a Scott well-order) of $\mathcal{A}_0$ and $\mathcal{A}_1$.

If T is a tree, $T + 1$ is the tree which is obtained from T by adding a new element at the end of every maximal branch of T. Note that $T + 1$ may be uncountable even if T is countable.

Lemma 9.82 *Suppose $S \subseteq \omega_1$ is bistationary, $\mathcal{A}_0 = \Phi(S)$, $\mathcal{A}_1 = \Phi(\emptyset)$, and $\mathcal{P} = T(\omega_1 \setminus S) + 1$. Then* **I** *wins* $\text{EF}_{\sigma\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$.

Proof Suppose **I** has already played $(c_\beta, x_\beta, p_\beta)$ and **II** has played y_β for $\beta < \alpha$. Suppose **I** now has to decide how to play $(c_\alpha, x_\alpha, p_\alpha)$ in $\text{EF}_\mathcal{P}(\mathcal{A}_0, \mathcal{A}_1)$. We assume that **I** has played in such a way that

1. $p_\beta = \langle\langle \delta_\delta : \delta \leq \gamma\rangle : \gamma < \beta\rangle$ ($\in \sigma(T(\omega_1 \setminus S) + 1)$.
2. $x_{\nu+2n} < y_{\nu+2n+1}$ in $\mathcal{A}_0$.
3. $x_{\nu+2n+1} < y_{\nu+2n+2}$ in $\mathcal{A}_1$.

9.6 Topology of Uncountable Models

Countable models with countable vocabulary can be thought of as points in the Baire space ω^ω. Likewise, models $\mathcal{M}$ of cardinality κ with vocabulary of cardinality κ can be thought of as points $f_\mathcal{M}$ in the set κ^κ. We can make κ^κ a topological space by letting the sets

$$N(f, \alpha) = \{g \in \omega^\kappa : f \restriction \alpha = g \restriction \alpha\},$$

where $\alpha < \kappa$, form the basis of the topology. Let us denote this *generalized Baire space* κ^κ by $\mathcal{N}_\kappa$. Now properties of models of size κ correspond to subsets of $\mathcal{N}_\kappa$. In particular, modulo coding, isomorphism of structures of cardinality κ becomes an "analytic" property in this space.

One of the basic questions about models of size κ that we can try to attack with methods of logic is the question which of those models can be identified up to isomorphism by means of a set of invariants. Shelah's Main Gap Theorem gives one answer: If $\mathcal{M}$ is any structure of cardinality $\kappa \geq \omega_1$ in a countable vocabulary, then the first-order theory of $\mathcal{M}$ is either of the two types:

Structure Case All uncountable models elementary equivalent to $\mathcal{M}$ can be characterized in terms of dimension-like invariants.

Non-structure Case In every uncountable cardinality there are non-isomorphic models elementary equivalent to $\mathcal{M}$ that are extremely difficult to distinguish from each other by means of invariants.

The game-theoretic methods we have developed in this book help us to analyze further the non-structure case. For this we need to develop some basic topology of $\mathcal{N}_\kappa$. A set $A \subseteq \mathcal{N}_\kappa$ is *dense* if A meets every non-empty open set. The space $\mathcal{N}_\kappa$ has a dense subset of size $\kappa^{<\kappa}$ consisting of all eventually constant functions. If the *Generalized Continuum Hypothesis GCH* is assumed, then $\kappa^{<\kappa} = \kappa$ for all regular κ and $\kappa^{<\kappa} = \kappa^+$ for singular κ.

Theorem 9.87 (Baire Category Theorem) *Suppose A_α, $\alpha < \kappa$, are dense open subsets of $\mathcal{N}_\kappa$. Then $\bigcap_\alpha A_\alpha$ is dense.*

Proof Let $f_0 \in \mathcal{N}_\kappa$ and $\alpha_0 < \kappa$ be arbitrary. If f_ξ and α_ξ for $\xi < \beta$ have been defined so that

$$\alpha_\zeta < \alpha_\xi \text{ and } f_\xi \in N(f_\zeta, \alpha_\zeta)$$

for $\zeta < \xi < \beta$, then we define f_β and α_β as follows: Choose some $g \in \mathcal{N}_\kappa$ such that $g \in N(f_\xi, \alpha_\xi)$ for all $\xi < \beta$ and let $\alpha_\beta = \sup_{\xi<\beta} \alpha_\xi$. Since A_β is dense, there is $f_\beta \in A_\beta \cap N(g, \alpha_\beta)$. When all f_ξ and α_ξ for $\xi < \kappa$ have been defined, we let f be such that $f \in N(f_\xi, \alpha_\xi)$ for all $\xi < \kappa$. Then $f \in \bigcap_\alpha A_\alpha \cap N(f_0, \alpha_0)$. □

Definition 9.88 A subset A of $\mathcal{N}_\kappa$ is said to be $\mathbf{\Sigma}^1_1$ (or *analytic*) if it is a projection of a closed subset of $\mathcal{N}_\kappa \times \mathcal{N}_\kappa$. A set is $\mathbf{\Pi}^1_1$ (or *co-analytic*) if its complement is analytic. Finally, a set is $\mathbf{\Delta}^1_1$ if it is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

Example 9.89 Examples of analytic sets relevant if κ is a regular cardinal $> \omega$, are

$$\mathrm{CUB}_\kappa = \{f \in \mathcal{N}_\kappa : \{\alpha < \kappa : f(\alpha) = 0\} \text{ contains a club}\}$$

and

$$\mathrm{NS}_\kappa = \{f \in \mathcal{N}_\kappa : \{\alpha < \kappa : f(\alpha) \neq 0\} \text{ contains a club}\}.$$

The set of α-sequences of elements of κ for various $\alpha < \kappa$ form a tree $\mathcal{N}_{<\kappa}$ under the subsequence relation. Any subset T of $\mathcal{N}_{<\kappa}$ which is closed under subsequences is called a *tree* in this section. A κ-branch of such a tree is any linear subtree (branch) of height κ. Let us denote $\langle g(\beta) : \beta < \alpha\rangle$ by $\bar{g}(\alpha)$.

Lemma 9.90 *A set $A \subseteq \mathcal{N}_\kappa$ is analytic iff there is a tree $T \subseteq \mathcal{N}_{<\kappa} \times \mathcal{N}_{<\kappa}$ such that for all f:*

$$f \in A \iff T(f) \text{ has a } \kappa\text{-branch}, \tag{9.7}$$

where $T(f) = \{\bar{g}(\alpha) : (\bar{g}(\alpha), \bar{f}(\alpha)) \in T\}$. Such a tree is called a tree representation *of A.*

Proof Suppose first A is analytic and $B \subseteq \kappa^\kappa \times \kappa^\kappa$ is a closed set such that

$$f \in A \iff \exists g((f, g) \in B).$$

Let

$$T = \{(\bar{f}(\alpha), \bar{g}(\alpha)) : (f, g) \in B, \alpha < \kappa\}.$$

Clearly now $f \in A$ if and only if $T(f)$ has a κ-branch. Conversely, suppose such a T exists. Let B be the set of (f, g) such that $(\bar{f}(\alpha), \bar{g}(\alpha)) \in T$ for all $\alpha < \kappa$. The set B is closed and its projection is A. □

Respectively, a set is co-analytic if and only if there is a tree $T \subseteq \mathcal{N}_{<\kappa} \times \mathcal{N}_{<\kappa}$ such that for all f:

$$f \in A \iff T(f) \text{ has no } \kappa\text{-branches}. \tag{9.8}$$

Let $\mathcal{T}_\kappa$ denote the class of all trees without κ-branches. Let $\mathcal{T}_{\lambda,\kappa}$ denote the set of subtrees of $\lambda^{<\kappa}$ of cardinality $\leq \lambda$ without any κ-branches.

Proposition 9.91 *Suppose B is a co-analytic subset of $\mathcal{N}_\kappa$ and T is as in (9.8). For any tree $S \in \mathcal{T}_\kappa$ let*

$$B_S = \{f \in B : T(f) \leq S\}.$$

Then

$$B = \bigcup_{S \in \mathcal{T}_{\lambda,\kappa}} B_S,$$

where $\lambda = \kappa^{<\kappa}$.

Proof Clearly $B_S \subseteq B$ if $S \in \mathcal{T}_\kappa$. Conversely, suppose $f \in B$. Then of course $f \in B_{T(f)}$. It remains to observe that $|T(f)| \leq \kappa^{<\kappa}$. □

Suppose $A \subseteq B$ is analytic and S is a tree as in (9.7). Let

$$T' = \{(\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha)) : \bar{g}(\alpha) \in T(f), \bar{h}(\alpha) \in S(f)\}. \tag{9.9}$$

Note that $|T'| \leq \kappa^{<\kappa}$ and T' has no κ-branches, for such a branch would give rise to a triple (f, g, h) which would satisfy $f \in A \setminus B$. Note also that if $f \in A$, then there is a κ-branch $\{\bar{h}(\alpha) : \alpha < \kappa\}$ in $S(f)$, and hence the mapping

$$\bar{g}(\alpha) \mapsto (\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha))$$

witnesses

$$T(f) \leq T'.$$

We have proved:

Proposition 9.92 (Covering Theorem for $\mathcal{N}_\kappa$) *Suppose B is a co-analytic subset of $\mathcal{N}_\kappa$ and S is as in (9.8). Suppose $A \subseteq B$ is analytic. Then*

$$A \subseteq B_T$$

for some $T \in \mathcal{T}_{\lambda,\kappa}$, where $\lambda = \kappa^{<\kappa}$.

The idea is that the sets B_T, $T \in \mathcal{T}_{\lambda,\kappa}$, cover the co-analytic set B completely, and moreover any analytic subset of B can be already covered by a single B_T. Especially if B happens to be $\mathbf{\Delta}^1_1$, then there is $T \in \mathcal{T}_{\lambda,\kappa}$ such that $B = B_T$.

Corollary (Souslin–Kleene Theorem for $\mathcal{N}_\kappa$) *Suppose B is a $\mathbf{\Delta}^1_1$ subset of $\mathcal{N}_\kappa$. Then*

$$B = B_T$$

for some $T \in \mathcal{T}_{\lambda,\kappa}$, where $\lambda = \kappa^{<\kappa}$.

Corollary (Luzin Separation Theorem for $\mathcal{N}_\kappa$) *Suppose A and B are disjoint analytic subsets of $\mathcal{N}_\kappa$. Then there is a set of the form C_T for some co-analytic set C and some $T \in \mathcal{T}_{\lambda,\kappa}$, where $\lambda = \kappa^{<\kappa}$, that separates A and B, i.e. $A \subseteq C$ and $C \cap B = \emptyset$.*

In the case of classical descriptive set theory, which corresponds to assuming $\kappa = \omega$, the sets B_T are Borel sets. If we assume CH, then CUB and NS cannot be separated by a Borel set.

Proposition 9.93 *If $\kappa^{<\kappa} = \kappa$, then the sets B_T are analytic. If in addition T is a strong bottleneck, then B_T is $\mathbf{\Delta}^1_1$.*

Let us call a family $\mathcal{B}$ of elements of $\mathcal{T}_{\lambda,\kappa}$ *universal* if for every $T \in \mathcal{T}$ there is some $S \in \mathcal{B}$ such that $T \leq S$. If $\mathcal{T}_{\lambda,\kappa}$ has a universal family of size μ, and $\kappa^{<\kappa} = \kappa$, then by the above results every co-analytic set in $\mathcal{N}_\kappa$ is the union of μ analytic sets. By results in Mekler and Väänänen (1993) it is consistent relative to the consistency of ZFC that $\mathcal{T}_{\kappa^+}$, $2^\kappa = \kappa^+$, has a universal family of size κ^{++} while $2^{\kappa^+} = \kappa^{+++}$.

Definition 9.94 The class of *Borel* subsets of $\mathcal{N}_\kappa$ is the smallest class containing the open sets and the closed sets which is closed under unions and intersections of length κ.

Note that every closed set in $\mathcal{N}_\kappa$ is the union of $\kappa^{<\kappa}$ open sets (Exercise 9.57). So if $\kappa^{<\kappa} = \kappa$, then the definition of Borelness can be simplified.

Theorem 9.95 *Assume $\kappa^{<\kappa} = \kappa > \omega$. Then $\mathcal{N}_\kappa$ has two disjoint analytic sets that cannot be separated by Borel sets.*

Proof Note that κ is a regular cardinal. Every Borel set A has a "Borel code" c such that $A = B_c$. Let us suppose $A = B_c$ separates the disjoint analytic sets $\mathcal{CUB}_\kappa$ and NS_κ defined in Example 9.89. For example, $\mathcal{CUB} \subseteq A$ and $A \cap \mathrm{NS}_\kappa = \emptyset$. Let $\mathcal{P} = (2^{<\kappa}, \leq)$ be the Cohen forcing for adding a generic subset for κ. Let G be $\mathcal{P}$-generic and $g = \bigcup \mathcal{P}$. Now either $g \in A$ or $g \notin A$. Let us assume, w.l.o.g., that $g \in A$. Let $p \Vdash \check{g} \in B_{\check{c}}$. Let $M \prec (H(\mu), \in, <^*)$ for a large μ such that $\kappa, p, \mathcal{P}, TC(c) \in M$, $M^{<\kappa} \subseteq M$, and $<^*$ is a well-order of $H(\mu)$. Since $\kappa^{<\kappa} = \kappa > \omega$, we may also assume $|M| = \kappa$. Since $\mathcal{P}$ is $< \kappa$-closed, it is easy to construct a $\mathcal{P}$-generic G' over M in V such that

$$\{\alpha < \kappa : M \models \text{“}(\check{g})_{G'}(\alpha) \neq 0\text{”}\} \text{ contains a club.} \tag{9.10}$$

It is easy to show that $B_c = (B_{\check{c}})_{G'}$. Since

$$M \models \text{“}p \Vdash \check{g} \in B_{\check{c}}\text{”},$$

whence $(\check{g})_{G'} \in B_c$ and therefore $(\check{g})_{G'} \notin \mathrm{NS}_\kappa$. This contradicts (9.10). □

Example 9.96 Suppose $\mathcal{M}$ is a structure with $M = \kappa$. We call the analytic set

$$\{\mathcal{N} : N = \kappa \text{ and } \mathcal{N} \cong \mathcal{M}\}$$

the *orbit* of $\mathcal{M}$. Let $\mathcal{N} \ncong \mathcal{M}$. Now player **I** has an obvious winning strategy ρ in $\mathrm{EF}_\kappa(\mathcal{M}, \mathcal{N})$: he simply makes sure that all elements of both models are played. Obviously there are many ways to play all the elements but any of them will do. Let us consider the co-anaytic set $B = \{f_\mathcal{N} : N = \kappa \text{ and } \mathcal{N} \ncong \mathcal{M}\}$. Let $S(\mathcal{N})$ be the Scott tree S_ρ of the pair $(\mathcal{M}, \mathcal{N})$. Let us choose a tree representation T of B in such a way that for all $\mathcal{N}$ with $N = \kappa$, $T(f_\mathcal{N}) = S(\mathcal{N})$. If now $f_\mathcal{N} \in B_{T'}$, then player **I** wins $\mathrm{EF}_{T'}(\mathcal{M}, \mathcal{N})$.

Recall that if $\mathcal{M}$ is a countable structure and α is the Scott height of $\mathcal{M}$, then **I** wins $\mathrm{EFD}_{\alpha+\omega}(\mathcal{M}, \mathcal{N})$ whenever $\mathcal{M} \ncong \mathcal{N}$ and N is countable. Equivalently, using the notation of Example 9.54, player **I** wins $\mathrm{EF}_{B_{\alpha+\omega}}(\mathcal{M}, \mathcal{N})$ whenever $\mathcal{M} \ncong \mathcal{N}$ and N is countable. We now generalize this property of $B_{\alpha+\omega}$ to uncountable structures.

Definition 9.97 Suppose κ is an infinite cardinal and $\mathcal{M}$ is a structure of cardinality κ. A tree T is a *universal Scott tree* of a structure $\mathcal{M}$ if T has no branches of length κ and player **I** wins $\mathrm{EF}_{\sigma T}(\mathcal{M}, \mathcal{N})$ whenever $\mathcal{M} \ncong \mathcal{N}$ and $|N| = |M|$.

The idea of the universal Scott tree is that the tree T alone suffices as a clock for player **I** to win all the 2^κ different games $\mathrm{EF}_T(\mathcal{M}, \mathcal{N})$ where $\mathcal{M} \ncong \mathcal{N}$ and $|N| = |M|$. Universal Scott trees exist: there is always a universal Scott tree of cardinality $\leq 2^\kappa$ as we can put the various Scott trees of the pairs $(\mathcal{M}, \mathcal{N})$, $\mathcal{M} \ncong \mathcal{N}$, $|M| = |N|$, each of them of the size $\leq \kappa^{<\kappa}$, together into one tree. So the question is: How small universal Scott trees does a given structure have?

If $\kappa^{<\kappa} = \lambda$ and $\mathcal{T}_{\lambda,\kappa}$ has a universal family of size μ, then every structure of size κ has a universal Scott tree of size μ.

If we allowed T to have a branch of length κ, any such tree would be a universal Scott tree of any structure of cardinality κ.

We ask whether **I** wins $\mathrm{EF}_{\sigma T}(\mathcal{M}, \mathcal{N})$ rather than in $\mathrm{EF}_T(\mathcal{M}, \mathcal{N})$ in order to preserve the analogy with the concept of a Scott tree. A universal Scott tree T in our sense would give rise to a universal Scott tree σT in the latter sense. Note that $|\sigma T| = |T|^{<\kappa}$, so this is the order of magnitude of a difference in the size of universal Scott trees in the two possible definitions.

Proposition 9.98 *Suppose $\kappa^{<\kappa} = \kappa$ and $\mathcal{M}$ is a structure with $M = \kappa$. The following are equivalent:*

(1) *The orbit of $\mathcal{M}$ is $\mathbf{\Delta}^1_1$.*

(2) $\mathcal{M}$ has a universal Scott tree of cardinality κ.

Proof Suppose first (2) is true. Then

$$\mathcal{M} \not\cong \mathcal{N} \iff \text{player } \mathbf{I} \text{ wins } \mathrm{EF}_{\sigma T}(\mathcal{M},\mathcal{N}).$$

The existence of a winning strategy of $\mathbf{I}$ can be written in $\mathbf{\Pi}^1_1$ form since we assume $\kappa^{<\kappa} = \kappa$. Assume then (1). Let ρ be a strategy of player $\mathbf{I}$ in $\mathrm{EF}_\kappa(\mathcal{M},\mathcal{N})$ in which he simply enumerates the universes. Note that this is independent of $\mathcal{N}$. Let $S(\mathcal{N})$ be the Scott tree S_ρ of the pair $(\mathcal{M},\mathcal{N})$. Let us consider the co-anaytic set $B = \{f_\mathcal{N} : N = \kappa \text{ and } \mathcal{N} \not\cong \mathcal{M}\}$. Let us choose a tree representation T of B as in Example 9.96. If now $f_\mathcal{N} \in B_{T'}$, then player $\mathbf{I}$ wins $\mathrm{EF}_{T'}(\mathcal{M},\mathcal{N})$. By the above Souslin–Kleene Theorem, (1) implies the existence of a tree T' such that $B = B'_T$. Thus for any $\mathcal{N}$ with $N = \kappa$, $\mathcal{M} \not\cong \mathcal{N}$ implies that player $\mathbf{I}$ wins $\mathrm{EF}_{T'}(\mathcal{M},\mathcal{N})$. Thus T' is a universal Scott tree of $\mathcal{M}$. Moreover, $|T'| = \kappa^{<\kappa} = \kappa$. □

The question whether the orbit of $\mathcal{M}$ is $\mathbf{\Delta}^1_1$ is actually highly connected to stability-theoretic properties of the first-order theory of $\mathcal{M}$, see Hyttinen and Tuuri (1991) for more on this.

9.7 Historical Remarks and References

Excellent sources for stronger infinitary languages are the textbook Dickmann (1975), the handbook chapter Dickmann (1985), and the book chapter Kueker (1975). The Ehrenfucht-Fraïssé Game for the logics $L_{\infty\lambda}$ appeared in Benda (1969) and Calais (1972). Proposition 9.32, Proposition 9.45, and the corollary of Proposition 9.45 are due to Chang (1968). The concept of Definition 9.40 and its basic properties were isolated independently by Dickmann (1975) and Kueker (1975). Theorem 9.31 is from Shelah (1990).

Looking at the origins of the transfinite Ehrenfeucht–Fraïssé Game, one can observe that the game plays a role in Shelah (1990), and is then systematically studied, first in the framework of back-and-forth sets in Karttunen (1984), and then explicitly as a game in Hyttinen (1987), Hyttinen (1990), Hyttinen and Väänänen (1990) and Oikkonen (1990).

The importance of trees in the study of the transfinite Ehrenfeucht–Fraïssé Game was first recognized in Karttunen (1984) and Hyttinen (1987). The crucial property of trees, or more generally partial orders, is Lemma 9.55 part (ii), which goes back to Kurepa (1956). A more systematic study of the quasi-order $\mathcal{P} \leq \mathcal{P}'$ of partial orders, with applications to games in mind, was started in Hyttinen and Väänänen (1990), where Lemma 9.57, Definition 9.58, Lemma 9.59, and Lemma 9.60 originate. The important role of the concept of persistency (Definition 9.63) gradually emerged and was explicitly isolated and exploited in Huuskonen (1995). Once it became clear that trees may be incomparable by $\leq$, the concept of bottleneck arose quite naturally. Definition 9.64 is from Todorčević and Väänänen (1999). The relative consistency of the non-existence of non-trivial bottlenecks (Theorem 9.65) was proved in Mekler and Väänänen (1993). For more on the structure of trees see Todorčević and Väänänen (1999) and Džamonja and Väänänen (2004).

The point of studying trees in connection with the transfinite Ehrenfeucht–Fraïssé Game is that there are two very natural tree structures behind the game. The first tree that arises from the game is the tree of sequences of moves, as in Lemma 9.73. This tree originates in Karttunen (1984). The second, and in a sense more powerful tree is the tree of strategies of a player, as in Definition 9.69 and the subsequent Proposition 9.71. This idea originates from Hyttinen (1987).

The "transfinite" analogues of Scott ranks are the Scott and Karp trees, introduced in Hyttinen and Väänänen (1990). Because of problems of incomparability of some trees, the picture of the "Scott watershed" is much more complicated than in the case of games of length ω, as one can see by comparing Figure 7.4 and Figure 9.7. Proposition 9.85 and Theorem 9.86 are from Tuuri (1990).

There is a form of infinitary logic the elementary equivalence of which corresponds exactly to the existence of a winning strategy for **II** in EF_α, in the spirit of the Strategic Balance of Logic. These infinitary logics are called *infinitely deep languages*. Their formulas are like formulas of $L_{\kappa\lambda}$ but there are infinite descending chains of subformulas. Thus, if we think of the syntax of a formula as a tree, the tree may have transfinite rank. These languages were introduced in Hintikka and Rantala (1976) and studied in Karttunen (1979), Rantala (1981), Karttunen (1984), Hyttinen (1990), and Tuuri (1992). See Väänänen (1995) for a survey on the topic.

There is also a transfinite version of the Model Existence Game, the other leg of the Strategic Balance of Logic, with applications to undefinability of (generalized) well-order and Separation Theorems, see Tuuri (1992) and Oikkonen (1997).

It was recognized already in Shelah (1990) that the roots of the problem of extending the Scott Isomorphism Theorem to uncountable cardinalities lie in stability theoretic properties of the models in question. This was made explicit in the context of transfinite Ehrenfeucht–Fraïssé Games in Hyttinen and Tuuri (1991). It turns out that there is indeed a close connection between the structure of Scott and Karp trees of elementary equivalent uncountable models and the stability theoretic properties such as superstability, DOP, and OTOP, of the (common) first-order theory. For more on this, see Hyttinen (1992), Hyttinen et al. (1993), and Hyttinen and Shelah (1999).

A good testing field for the power of long Ehrenfeucht–Fraïssé Games turned out to be the area of almost free groups, where it seemed that the applicability of the infinitary languages $L_{\kappa\lambda}$ had been exhausted. For results in this direction, see Mekler and Oikkonen (1993), Eklof et al. (1995), Shelah and Väisänen (2002), and Väisänen (2003).

An alternative to considering transfinite Ehrenfeucht–Fraïssé Games is to study isomorphism in a forcing extension. Isomorphism in a forcing extension is called potential isomorphism. The basic reference is Nadel and Stavi (1978). See also Huuskonen et al. (2004).

Early on it was recognized that the trees $T(S)$ (see Example 9.61) are very useful and in some sense fundamental in the area of transfinite Ehrenfeucht–Fraïssé Games. The question arose, whether there is a largest such tree for $S \subseteq \omega_1$ bistationary. Quite unexpectedly the existence of a largest such tree turned out to be consistent relative to the consistency of ZF. The name "Canary trees" was coined for them, because such a tree would indicate whether some stationary set was killed. See Mekler and Shelah (1993) and Hyttinen and Rautila (2001) for results on the Canary tree.

While the Ehrenfeucht–Fraïssé Game of length ω is almost trivially determined, the Ehrenfeucht–Fraïssé Game of length ω_1 (and also of length $\omega + 1$) can be non-determined, see Hyttinen (1992), Mekler et al. (1993), and Hyttinen et al. (2002). This has devastating consequences for attempts to use transfinite Ehrenfeucht–Fraïssé Games to classify uncountable models. It is a phenomenon closely related to the incomparability of non-well-founded trees by the relation $\leq$. This non-determinism is ultimately also the reason why the simple picture in Figure 7.4 becomes Figure 9.7.

Some of the complexities of uncountable models can be located already on the topological level, as is revealed by the study of the spaces $\mathcal{N}_\kappa$. These spaces were studied under the name of κ-metric spaces in Sikorski (1950), Juhász and Weiss (1978), and Todorčević (1981b). Their role as spaces of models, in the spirit of Vaught (1973), was emphasized in Mekler and Väänänen (1993). For more on the topology of uncountable models, see Väänänen (1991), Väänänen (1995), and Shelah and Väänänen (2000). See Väänänen (2008) for an informal exposition of some basic ideas. Theorem 9.95 is from Shelah and Väänänen (2000).

Exercise 9.22 is from Nadel and Stavi (1978). Exercises 9.29 and 9.30 are from Hyttinen (1987). Exercise 9.35 is from Hyttinen and Väänänen (1990).

Exercise 9.40 is from Kurepa (1956). Exercise 9.41 is from Huuskonen (1995). Exercise 9.47 is from Todorčević (1981a). Exercise 9.56 is due to Lauri Hella.

Exercises

9.1 Show that player **II** wins $\mathrm{EF}_{\omega}^{\aleph_0}(\mathcal{M}, \mathcal{M}')$ if and only if she has a winning strategy in $\mathrm{EF}_{\omega}(\mathcal{M}, \mathcal{M}')$.

9.2 Show that **I** wins $\mathrm{EFD}_2^{\omega_1}(\mathcal{M}, \mathcal{N})$ if $\mathcal{M} = (\mathbb{Q}, <)$ and $\mathcal{N} = (\mathbb{R}, <)$.

9.3 Show that in Example 9.2 player **I** has a winning strategy already in $\mathrm{EFD}_2^{\omega_1}(\mathcal{M}, \mathcal{M}')$.

9.4 Show that $\mathcal{M} \simeq_p \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are as in Example 9.4.

9.5 Prove the claim of Example 9.5.

9.6 Prove the claim of Example 9.19.

9.7 Give necessary and sufficient conditions for player **I** to have a winning strategy in $\mathrm{EFD}_{\alpha}^{\kappa}(\mathcal{M}, \mathcal{M}')$, when $\mathcal{M}$ and $\mathcal{M}'$ are L-structures for a unary vocabulary L.

9.8 Show that if $\mathcal{M} = (M, d, \mathbb{R}, <_{\mathbb{R}})$ and $\mathcal{M}' = (M', d', \mathbb{R}, <_{\mathbb{R}})$ are separable metric spaces so that **II** has a winning strategy in $\mathrm{EFD}_3^{\omega_1}(\mathcal{M}, \mathcal{M}')$, then $\mathcal{M}$ is complete if and only if $\mathcal{M}'$ is.

9.9 Prove that any model class which is closed under isomorphisms and has only models of cardinality $\leq \lambda$ for some λ is definable in $L_{\infty\infty}$.

9.10 Fix λ and a vocabulary L. Prove that for every α there is only a set of logically non-equivalent formulas of $L_{\infty\lambda}$ of the vocabulary L and of quantifier rank $\leq \alpha$.

9.11 Prove that $\simeq_{\lambda}$ is an equivalence relation on $\mathrm{Str}(L)$ for any vocabulary L.

9.12 Suppose $\mathrm{cf}(\kappa) = \omega$ (i.e. $\kappa = \sup_n \kappa_n$, where $\kappa_0 < \kappa_1 < \cdots$). Show that $\mathcal{A} \simeq_{\kappa} \mathcal{B}$ implies $\mathcal{A} \cong \mathcal{B}$ if $|A| = |B| = \kappa$.

9.13 Suppose $\{\mathcal{A}_i : i \in I\}$ is a family of L-structures for a relational vocabulary L. Suppose furthermore $A_i \cap A_j = \emptyset$ for $i \neq j$. The *disjoint sum* of the family $\{\mathcal{A}_i : i \in I\}$ is the L-structure:

$$\biguplus_{i\in I} \mathcal{A}_i = \left(\bigcup_{i\in I} A_i, \left(\bigcup_{i\in I} R^{\mathcal{A}_i} \right)_{R\in L} \right).$$

Show that if $\{\mathcal{A}_i : i \in I\}$ and $\{\mathcal{B}_i : i \in I\}$ are families of L-structures for a relational vocabulary L and for each i

$$\mathcal{A}_i \simeq_{\lambda} \mathcal{B}_i,$$

then

$$\biguplus_{i\in I}\mathcal{A}_i \simeq_\lambda \biguplus_{i\in I}\mathcal{B}_i.$$

9.14 Suppose $\{\mathcal{A}_i : i \in I\}$ is a family of L-structures. The *direct product* of the family $\{\mathcal{A}_i : i \in I\}$ is the L-structure

$$\prod_{i\in I}\mathcal{A}_i = \left(\prod_{i\in I}A_i, \left(\prod_{i\in I}R^{\mathcal{A}_i}\right)_{R\in L}, (\mathrm{prod}_{i\in I}f^{\mathcal{A}_i})_{f\in L}, ((c : i\in I))_{c\in L}\right)$$

where

$$(\mathrm{prod}_{i\in I}f^{\mathcal{A}_i})((a : i\in I)) = (f_i^{\mathcal{A}_i}(a_i) : i \in I).$$

Show that if $\{\mathcal{A}_i : i \in I\}$ and $\{\mathcal{B}_i : i \in I\}$ are families of L-structures and for each $i \in I$

$$\mathcal{A}_i \simeq_\lambda \mathcal{B}_i,$$

then

$$\prod_{i\in I}\mathcal{A}_i \simeq_\lambda \prod_{i\in I}\mathcal{B}_i.$$

9.15 Suppose $\{\mathcal{A}_i : i \in I\}$ is a family of L-structures in a vocabulary L containing a distinguished constant symbol 0_L. The direct sum $\bigoplus_{i\in I}\mathcal{A}_i$ of the family $\{\mathcal{A}_i : i \in I\}$ is the substructure of $\prod_{i\in I}\mathcal{A}_i$ consisting of $(a_i : i \in I)$ such that $a_i = 0_L^{\mathcal{A}_i}$ for all but finitely many $i \in I$. Show that if $\{\mathcal{A}_i : i \in I\}$ and $\{\mathcal{B}_i : i \in I\}$ are such families and for all $i \in I$

$$\mathcal{A}_i \simeq_\lambda \mathcal{B}_i,$$

then

$$\bigoplus_{i\in I}\mathcal{A}_i \simeq_\lambda \bigoplus_{i\in I}\mathcal{B}_i.$$

9.16 Suppose $\mathcal{M}$ is an L-structure for a relational vocabulary L. Let $I \subseteq J$ be sets of size $\geq \lambda$. Show that

$$\bigoplus_{i\in I}\mathcal{M} \simeq_\lambda \bigoplus_{i\in J}\mathcal{M}.$$

9.17 Consider $\mathbb{Z}$ as an abelian group. Show that for any set I:

$$\bigoplus_{i\in I}\mathbb{Z} \simeq_p \prod_{i\in I}\mathbb{Z}.$$

9.18 Show that "has a clique of size λ" is not definable in $L_{\infty\lambda}$.

9.19 Prove Exercise 9.13 for $\equiv^{\alpha}_{\infty\omega}$.

9.20 Prove Theorem 9.29.

9.21 Prove Proposition 9.32.

9.22 Let us write $\mathcal{M}(\lambda - \mathrm{PI})\mathcal{N}$ if there is a forcing notion which does not add new sets of cardinality $< \lambda$ (such forcing is called $< \lambda$-distributive) which forces $\mathcal{M}$ and $\mathcal{N}$ to be isomorphic. This is a form of "potential isomorphism", i.e. isomorphism in a forcing extension. Show that $\mathcal{M}(\lambda - \mathrm{PI})\mathcal{N}$ is not a transitive relation among structures and thereby does not correspond to elementary equivalence relative to any logic. (Hint: Use the models $\Phi(A)$ of Definition 9.8.)

9.23 Show that if $\mathcal{M}$ is λ-homogeneous, then for any sequences $\vec{a}$ and $\vec{b}$ of the same length from M:

$$(\mathcal{M}, \vec{a}) \equiv (\mathcal{M}, \vec{b}) \Rightarrow (\mathcal{M}, \vec{a}) \simeq^s_\lambda (\mathcal{M}, \vec{b}).$$

9.24 Let H_α be the lexicographically ordered set of sequences $s \in {}^{\omega_\alpha}\{0,1\}$ (i.e. $s <_{H_\alpha} s'$ if $s(\xi) < s'(\xi)$ for the least ξ such that $s(\xi) \neq s'(\xi)$) for which there is a $\beta < \omega_\alpha$ such that $s(\beta) = 1$ and $s(\gamma) = 0$ for $\beta \leq \gamma < \omega_\alpha$. Show that $H_{\alpha+1}$ is an $\eta_{\alpha+1}$-set.

9.25 Show that H_α is an η_α-set if and only if $\aleph_\alpha$ is regular.

9.26 Show that any η_α-set for singular $\aleph_\alpha$ is also an $\eta_{\alpha+1}$-set.

9.27 Prove that if $\mathcal{A}$ and $\mathcal{B}$ are η_α-sets, then $\mathcal{A} \simeq_{\aleph_\alpha} \mathcal{B}$, and if moreover $\aleph_\alpha$ is regular, then $\mathcal{A} \simeq^s_{\aleph_\alpha} \mathcal{B}$.

9.28 Suppose $\mathcal{M}$ and $\mathcal{M}'$ are real-closed fields whose underlying orders are η_α sets. Show that $\mathcal{M} \simeq_{\aleph_\alpha} \mathcal{M}'$ and if moreover $\aleph_\alpha$ is regular, then $\mathcal{M} \simeq^s_{\aleph_\alpha} \mathcal{M}'$.

9.29 Suppose $S \subseteq \omega_1$. Show that S contains a cub if and only if **I** wins the game $\mathrm{EF}_{\omega+2}(\Phi(S), \Phi(\emptyset))$. (Hint: It is a good idea to consider for a given strategy the set of ordinals $< \omega_1$ which are in some appropriate sense "closed under the first ω moves of the strategy".)

9.30 Show that $S \subseteq \omega_1$ is disjoint from a cub if and only if **II** wins the game $\mathrm{EF}_{\omega+2}(\Phi(S), \Phi(\emptyset))$. (Hint: It is a good idea to consider for a given strategy the set of ordinals $< \omega_1$ which are in some appropriate sense "closed under the first ω moves of the strategy".)

9.31 Show that if $\mathcal{M} \simeq^s_{\aleph_1} \mathcal{N}$, then **II** wins the game $\mathrm{EF}_{\omega_1}(\mathcal{M}, \mathcal{N})$.

9.32 Show that **II** wins the game $\mathrm{EF}_{\omega_1}(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M}$ and $\mathcal{N}$ are potentially isomorphic in the following sense: there is a countably closed[5] forcing notion $\mathcal{P}$ such that $\mathcal{P}$ forces $\mathcal{M} \cong \mathcal{N}$. (Hint: Note that the forcing which collapses $|M \cup N|$ to $\aleph_1$ is countably closed.)

[5] I.e. every countable descending chain of conditions has a lower bound.

9.33 Show that $(\omega_1, <)$ and $(\mathbb{R}, <)$ are incomparable by the quasi-order $\leq$ of po-sets.

9.34 Prove $\sigma\mathcal{P} \leq \sigma\mathcal{P}' \iff \sigma'\mathcal{P} \leq \mathcal{P}'$.

9.35 Suppose $S \subseteq \omega_1$. Let $T(S)$ be the tree of closed ascending sequences of elements of S. Choose disjoint bistationary sets S_1 and S_2. Then $T(S_1) \not\leq T(S_2)$ and $T(S_2) \not\leq T(S_1)$ (see Example 9.61).

9.36 Prove the claims of Example 9.54.

9.37 Suppose T is a tree. Show that T has no infinite branches if and only if there is an ordinal α so that $T \equiv (\alpha, >)$.

9.38 Prove that if a po-set $\mathcal{P}$ is a union of countably many antichains, it satisfies $\mathcal{P} \leq (\mathbb{Q}, <)$.

9.39 Show that $F_{\aleph_1}$ and $T_p^{\aleph_1}$ are special trees.

9.40 Prove the claim in Example 9.56 that $\sigma'\mathbb{Q}$ is special but $\sigma\mathbb{Q}$ non-special.

9.41 Show that T_p^κ is the $\leq$-smallest persistent tree in $\mathcal{T}_\kappa$.

9.42 Prove that T_p^κ is a strong bottleneck in the class $\mathcal{T}_\kappa$

9.43 If T_i, $i \in I$, is a family of trees, let $\bigoplus_{i\in I} T_i$ be the tree which consists of a union of disjoint copies of T_i, $i \in I$, identified at the root. Show that $\bigoplus_{i\in I} T_i$ is the supremum of $\{T_i : i \in I\}$ in the sense that $T_i \leq \bigoplus_{i\in I} T_i$ for all $i \in I$ and if $T_i \leq T$ for all $i \in I$, then $\bigoplus_{i\in I} T_i \leq T$.

9.44 If T_i, $i \in I$, is a family of trees, let $\prod_{i\in I} T_i$ be the product tree

$$\prod_{i\in I} T_i = \{s : \mathrm{dom}(s) = I, \forall i \in I(s(i) \in T_i)\}.$$

$$s \leq s' \iff \forall i \in I(s(i) \leq_{T_i} s'(i)).$$

Let $\bigotimes_{i\in I} T_i$ be the subtree

$$\bigotimes_{i\in I} T_i = \left\{ s \in \prod_{i\in I} T_i : \forall i \in I \forall j \in I(\mathrm{ht}_{T_i}(s(i)) = \mathrm{ht}_{T_j}(s(j))\right\}.$$

We denote $\bigotimes_{i\in\{0,1\}} T_i$ by $T_0 \otimes T_1$. Prove that $\bigotimes_{i\in I} T_i$ is the infimum of $\{T_i' : i \in I\}$, that is, $\bigotimes_{i\in I} T_i \leq T_i$ for each $i \in I$, and if $T \leq T_i$ for all $i \in I$, then $T \leq \bigotimes_{i\in I} T_i$.

9.45 Show that $\mathcal{P}_\alpha \star \mathcal{P}'$ in the proof of Theorem 9.65 contains a κ-closed dense set. (Hint: Suppose $(s, s') \in \mathcal{P}_\alpha \star \mathcal{P}'$. Thus $s : \kappa \to \{0, 1\}$, $|s| < \kappa$, and s forces that s' is a closed sequence of length $< \kappa$ in A_α. Consider the sets of (s, s') for which $\sup\{\beta : s(\beta) = 1\} = \max(s')$.)

9.46 Suppose A and B are disjoint stationary subsets of a regular cardinal κ. Show that $T(A) \otimes T(B) \leq T_p^\kappa$. (Hint: Show that **II** has a winning strategy in the game $G(T(A) \otimes T(B), T_p^\kappa)$.)

9.47 Suppose $S \subseteq \omega_1$. Prove that $T(S) \leq \mathbb{Q}$ if and only if S is non-stationary.

9.48 Suppose $S \subseteq \omega_1$ is bistationary and T is Aronszajn. Show that $T(S) \not\leq T$ and $T \not\leq T(S)$.

9.49 Prove $\mathrm{b}((\mathbb{Q}, <)) = \mathrm{b}((\mathbb{R}, <)) = \omega_1$.

9.50 Prove Lemma 9.67.

9.51 Prove Lemma 9.83.

9.52 Show that if $\mathbf{I}$ wins $\mathrm{EF}_{T_i}(\mathcal{A}_0, \mathcal{A}_1)$ for all $i \in I$, then $\mathbf{II}$ does not win $\mathrm{EF}_{\bigotimes_{i\in I} T_i}(\mathcal{A}_0, \mathcal{A}_1)$.

9.53 Show that the family of Scott trees of $(\mathcal{A}_0, \mathcal{A}_1)$ is closed under suprema.

9.54 Show that the family of Karp trees of $(\mathcal{A}_0, \mathcal{A}_1)$ is closed under suprema.

9.55 Suppose $\mathcal{P}$ is a Scott po-set of $(\mathcal{A}_0, \mathcal{A}_1)$, where $\mathrm{Card}(A_0), \mathrm{Card}(A_1) \leq 2^{\aleph_0}$. Show that there is a Scott tree of $(\mathcal{A}_0, \mathcal{A}_1)$ such that $T \leq \mathcal{P}$ and $\mathrm{Card}(T) \leq 2^{\aleph_0}$.

9.56 Show that if $2^\kappa = 2^{\kappa^+}$, then $\mathcal{T}_{\kappa^+,\kappa^+}$ has an upper bound in $\mathcal{T}_{2^\kappa,\kappa^+}$.

9.57 Show that every closed set in $\mathcal{N}_\kappa$ is the union of $\kappa^{<\kappa}$ open sets.

9.58 Show that if $\mathrm{cf}(\kappa)\omega$, then the intersection of countably many open sets in $\mathcal{N}_\kappa$ is again open. Topological spaces with this property are called *σ-additive*.

9.59 Show $\mathcal{N}_\kappa$ has a basis consisting of clopen sets. Topological spaces with this property are called *zero-dimensional*.

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