Dependence Logic

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A possible new focus in logic

- Traditional logic is about truth values of sentences:
  - Valid, contingent, independent, possible, necessay, known, publicly announced, believed, etc.
- A possible new focus is on values of variables:
  - Constant value, non-constant value, functionally dependent value, independent from another, publicly announced value, believed value, etc.

A tool for focusing on values of variables

- Team = a set of assignments.
- Multiplicity.
- Collective action.
- Parallel action.
- Co-operation.

Team semantics

- Teams accomplish tasks by
  - Every member doing the same.
  - Dividing into subteams (skills).
  - Supplementing a new feature, (a skill).
  - Duplicating along a feature, (gender).
- Teams manifest dependence by e.g.
  - Having rank determine salary.
- Independence, e.g.
  - Having salary independent of gender.
  - Having time of descent independent of weight.

Case study of the new focus in logic

- Dependence logic
- *Dependence logic*, Cambridge University Press 2007.
- See Wikipedia entry on “dependence logic”.

Basic concept: dependence atom

=(x,y,z)

”z depends at most on x and y”

” x and y determine z”

”To know z, it suffices to know x and y”

$=(x_0,\ldots,x_n,z)$

Dependence atoms =(x,y,z)

+

First order logic

=

Dependence logic

Teams

- A team is just a *set* of assignments for a model.
- Special cases:
  - Empty team ∅.
    - Database with no rows.
  - The team {∅} with the empty assignment.
    - Database with no columns, and hence with at most one row.

Dependence logic *D*

$$t = t', \quad Rt_1 ... t_n$$

$$=\!(t_1, ..., t_n)$$

$$\varphi \vee \psi, \neg\varphi, \exists x_n \varphi$$

A team satisfies an identity $t=t'$ if every team member satisfies it.

	x_0	x_1	x_2
s_0	0	0	0
s_1	0	1	1
s_2	2	5	5

$$\mathfrak{M} \models_X t_1 = t_2 \quad \textit{iff} \quad \forall s \in X(t_1^{\mathfrak{M}}\langle s \rangle = t_2^{\mathfrak{M}}\langle s \rangle)$$

$$\mathfrak{M} \models_X t_1 \neq t_2 \quad \textit{iff} \quad \forall s \in X(t_1^{\mathfrak{M}}\langle s \rangle \neq t_2^{\mathfrak{M}}\langle s \rangle)$$

A team satisfies a relation $Rt_1 \ldots t_n$ if every team member does.

A team satisfies a relation $\neg Rt_1 \ldots t_n$ if every team member does.

	x_0	x_1	x_2
s_0	0	0	0
s_1	0	1	1
s_2	2	5	5

- A team X satisfies =(x,y,z) if in any two assignments in X, in which x and y have the same values, also z has the same value.

- A team X satisfies ¬=(x,y,z) if it is empty.

	x	y	u	z
s_0	0	0	1	0
s_1	0	1	0	2
s_2	2	5	0	5
s_3	0	1	1	2

$$\mathfrak{M} \models_X =\!(t_1, \ldots, t_n)$$

$$\forall s, s' \in X(t_1^{\mathfrak{M}}\langle s\rangle \neq t_1^{\mathfrak{M}}\langle s'\rangle \;\; or \\ \ldots \;\; or\; t_{n-1}^{\mathfrak{M}}\langle s\rangle \neq t_{n-1}^{\mathfrak{M}}\langle s'\rangle \; or\; t_n^{\mathfrak{M}}\langle s\rangle = t_n^{\mathfrak{M}}\langle s'\rangle)$$

An extreme case

=(x)

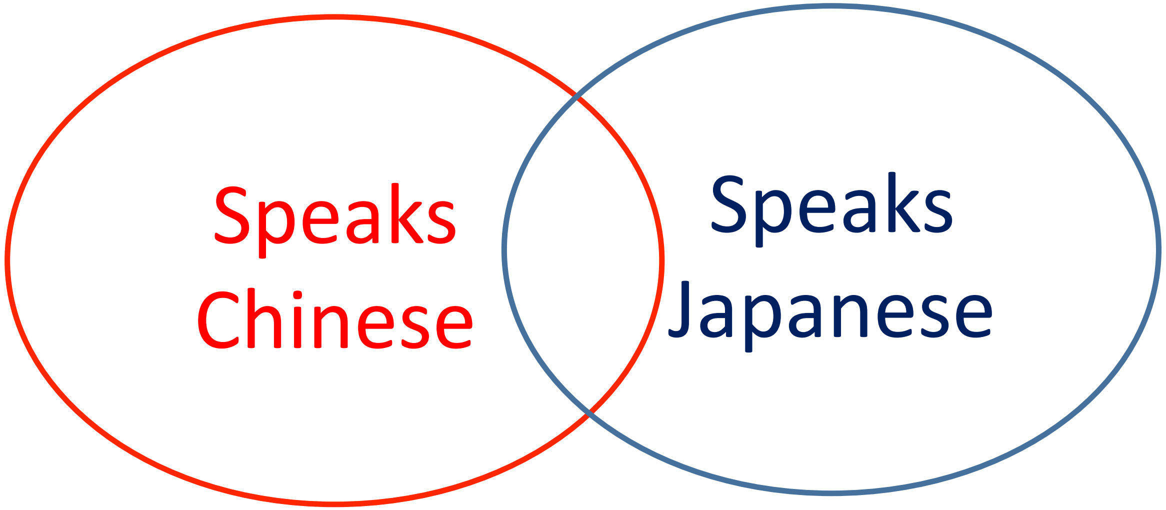
”x is constant in the team”

record	A1	A2	A3	A4	A5	A6
100000	8	6	7	3	0	6
100002	7	5	6	3	0	6
100003	4	8	7	3	0	6
100004	6	5	4	3	0	6
100005	6	12	65	3	0	6
100006	5	56	9	3	0	6
100007	6	23	0	4	0	8
...	...	...	...	...	...	...
408261	77	2	11	1	0	2

$$\mathfrak{M} \models_X \phi \vee \psi$$

there are X_0 *and* X_1 *such that* $\mathfrak{M} \models_{X_0} \phi$, $\mathfrak{M} \models_{X_1} \psi$, *and* $X \subseteq X_0 \cup X_1$

A team of Chinese **or** Japanese speakers:



Shorthands

$$\phi \wedge \psi \qquad \neg(\neg\phi \vee \neg\psi)$$

$$(\phi \rightarrow \psi) \qquad (\neg\phi \vee \psi)$$

$$(\phi \leftrightarrow \psi) \qquad ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$$

$$\forall x_n \phi \qquad \neg\exists x_n \neg\phi$$

$$\mathfrak{M} \models_X \phi \wedge \psi$$

$$both \quad \mathfrak{M} \models_X \phi \quad and \quad \mathfrak{M} \models_X \psi$$

$$\mathfrak{M} \models_X \exists x \phi$$

there is Y *such that* $\mathfrak{M} \models_Y \phi$ *and for every* $s \in X$ *we have* $s[a/x] \in Y$ *for some* $a \in M$

Team X can be supplemented with values for x so that φ becomes true

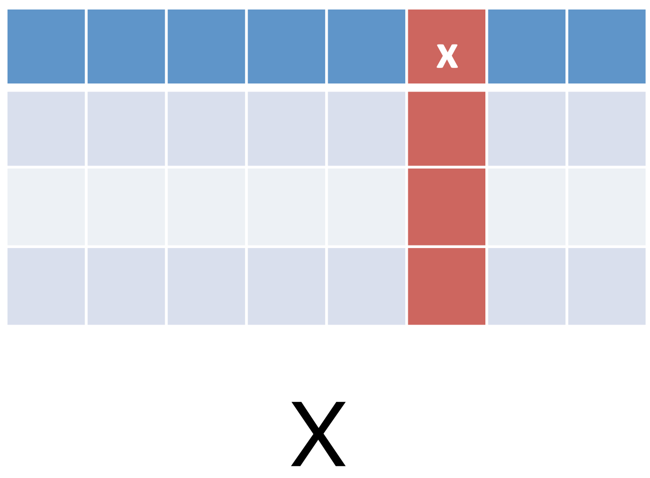
					x		

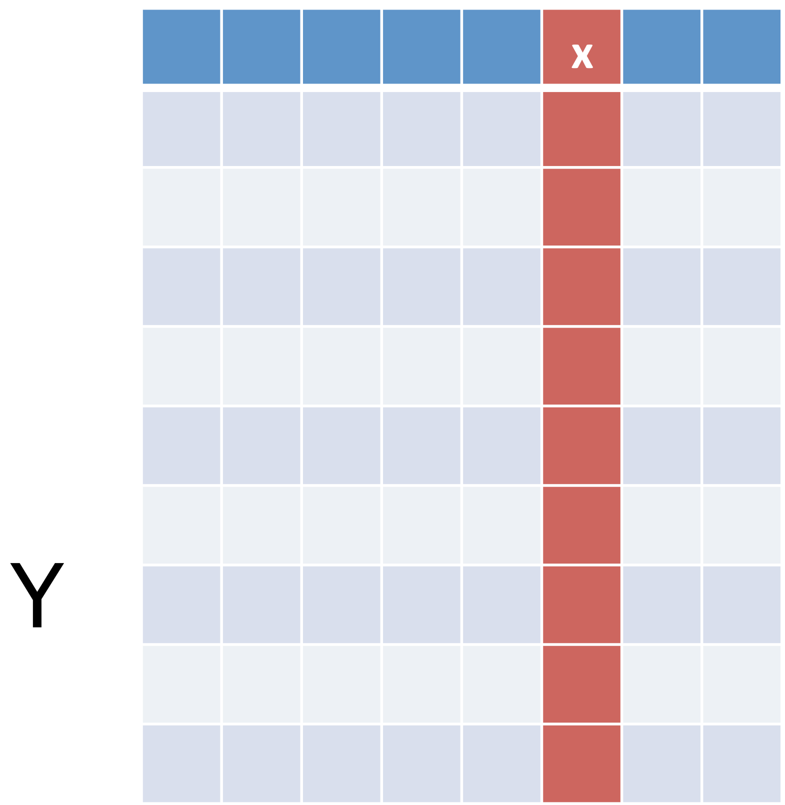
	u	x	w
X	Finnish Swedish Norwegian		driver author skier
Y	Finnish Swedish Norwegian	male female female	driver author skier

$$\mathfrak{M} \models_X \forall x \phi$$

there is Y *such that* $\mathfrak{M} \models_Y \phi$ *and for every* $s \in X$ *we have* $s[a/x] \in Y$ *for every* $a \in M$

Team X can be duplicated along x, by giving x all possible values, so that φ becomes true





	u	x	w
X	Finnish		driver
	Swedish		author
	Norwegian		skier
Y	Finnish	male	driver
	Finnish	female	driver
	Swedish	male	author
	Swedish	female	author
	Norwegian	male	skier
	Norwegian	female	skier

Logical consequence and equivalence

ψ follows logically from φ

$\phi \Rightarrow \psi$ $\mathcal{M} \models_X \phi$ implies $\mathcal{M} \models_X \psi$

ψ is logically equivalent with φ

$$\phi \equiv \psi,\ if\ \phi \Rightarrow \psi\ and\ \psi \Rightarrow \phi$$

Armstrong's rules

Always $=(x,x)$

If $=(x,y,z)$, then $=(y,x,z)$.

If $=(x,x,y)$, then $=(x,y)$.

If $=(x,z)$, then $=(x,y,z)$.

If $=(x,y)$ and $=(y,z)$, then $=(x,z)$.

Propositional rules

- From $\varphi \wedge \psi$ follows $\psi \wedge \varphi$.
- From $\varphi \vee \psi$ follows $\psi \vee \varphi$.
- From $\varphi \wedge (\psi \wedge \theta)$ follows $(\varphi \wedge \psi) \wedge \theta$.
- From $\varphi \vee (\psi \vee \theta)$ follows $(\varphi \vee \psi) \vee \theta$.
- From$(\varphi \vee \eta) \wedge (\psi \vee \theta)$ follows $(\varphi \wedge \psi) \vee (\varphi \wedge \theta) \vee (\eta \wedge \psi) \vee (\eta \wedge \theta)$.
- From$(\varphi \wedge \eta) \vee (\psi \wedge \theta)$ follows $(\varphi \vee \psi) \wedge (\varphi \vee \theta) \wedge (\eta \vee \psi) \wedge (\eta \vee \theta)$.
- From φ and ψ follows $\varphi \wedge \psi$.
- From $\varphi \wedge \psi$ follows φ.
- From φ follows $\varphi \vee \psi$.

Commutative

Associative

"Almost" distributive

Incorrect rules

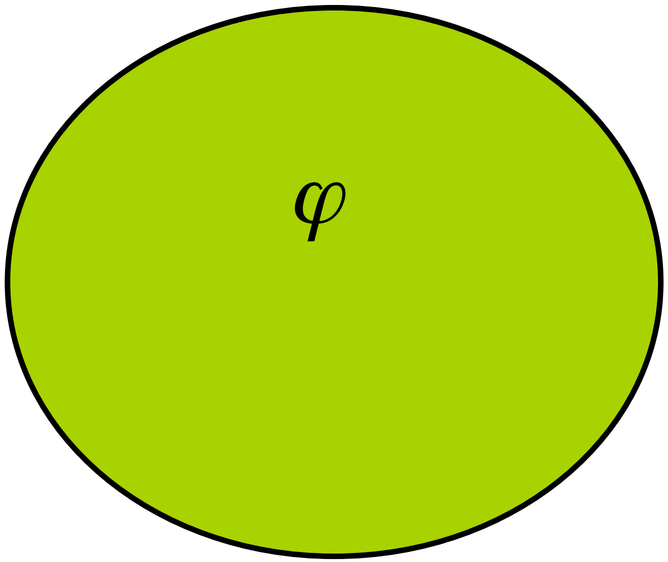
No absorption

- From $\varphi \vee \varphi$ follows φ. Wrong!
- From $(\varphi \wedge \psi) \vee (\varphi \wedge \theta)$ follows $\varphi \wedge (\psi \vee \theta)$. Wrong!
- From $(\varphi \vee \psi) \wedge (\varphi \vee \theta)$ follows $\varphi \vee (\psi \wedge \theta)$. Wrong!

Non-distributive

Example

- If $\varphi \rightarrow \psi$ is valid then φ *logically implies* ψ.



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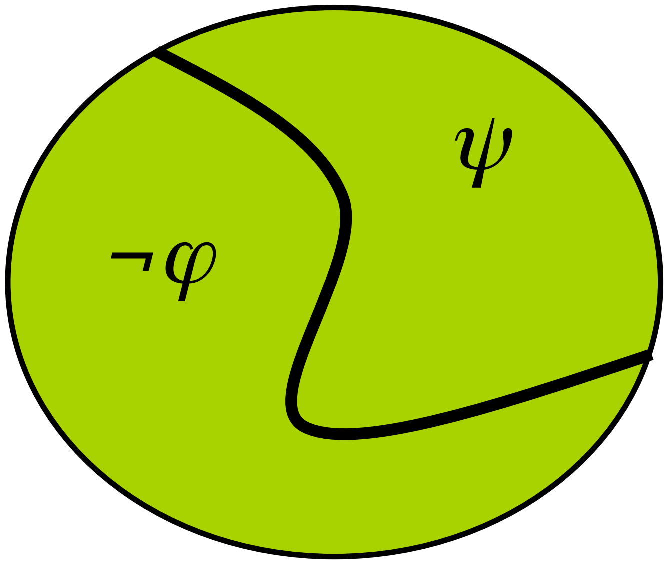

Example

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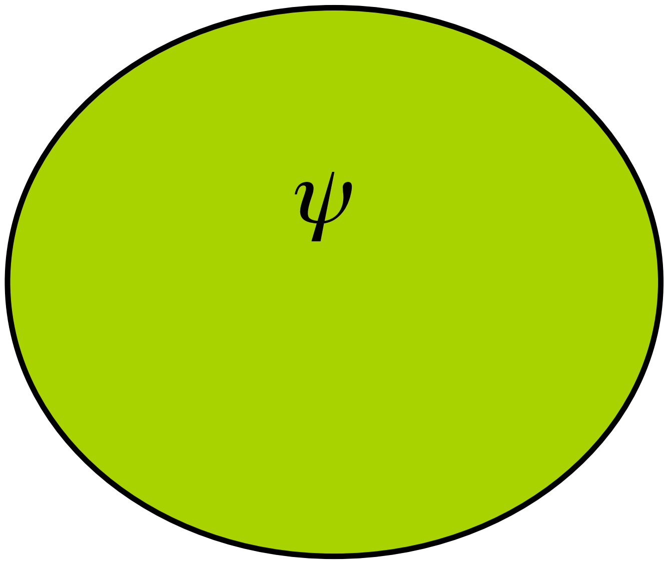


Quantifier rules

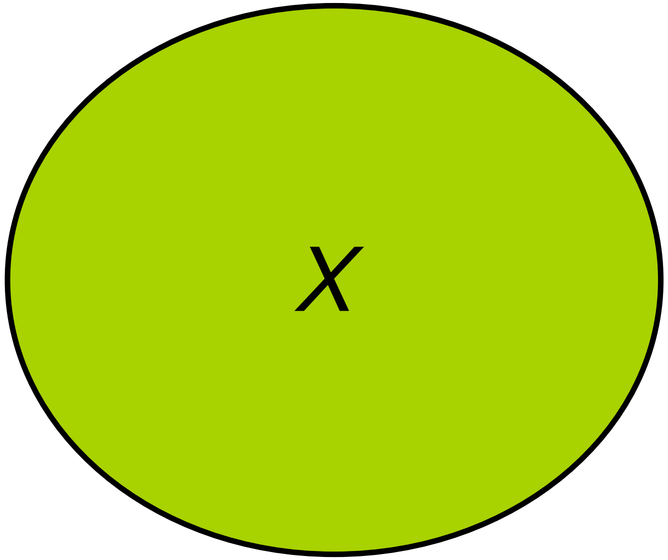
- From $\forall x\varphi \wedge \forall x\psi$ follows $\forall x(\varphi \wedge \psi)$, and vice versa.
- From $\exists x\varphi \vee \exists x\psi$ follows $\exists x(\varphi \vee \psi)$, and vice versa.
- From $\varphi \vee \forall x\psi$ follows $\forall x(\varphi \vee \psi)$, and vice versa, provided that x is not free in φ.
- From $\varphi \wedge \exists x\psi$ follows $\exists x(\varphi \wedge \psi)$, and vice versa, provided that x is not free in φ.
- From $\forall x\forall y\varphi$ follows $\forall y\forall x\varphi$.
- From $\exists x\exists y\varphi$ follows $\exists y\exists x\varphi$.
- From φ follows $\exists x\varphi$.
- From $\forall x\varphi$ follows φ.

Universal generalization

If $\varphi \rightarrow \psi$ is valid and x is not free in φ, then $\varphi \rightarrow \forall x \psi$ is valid.

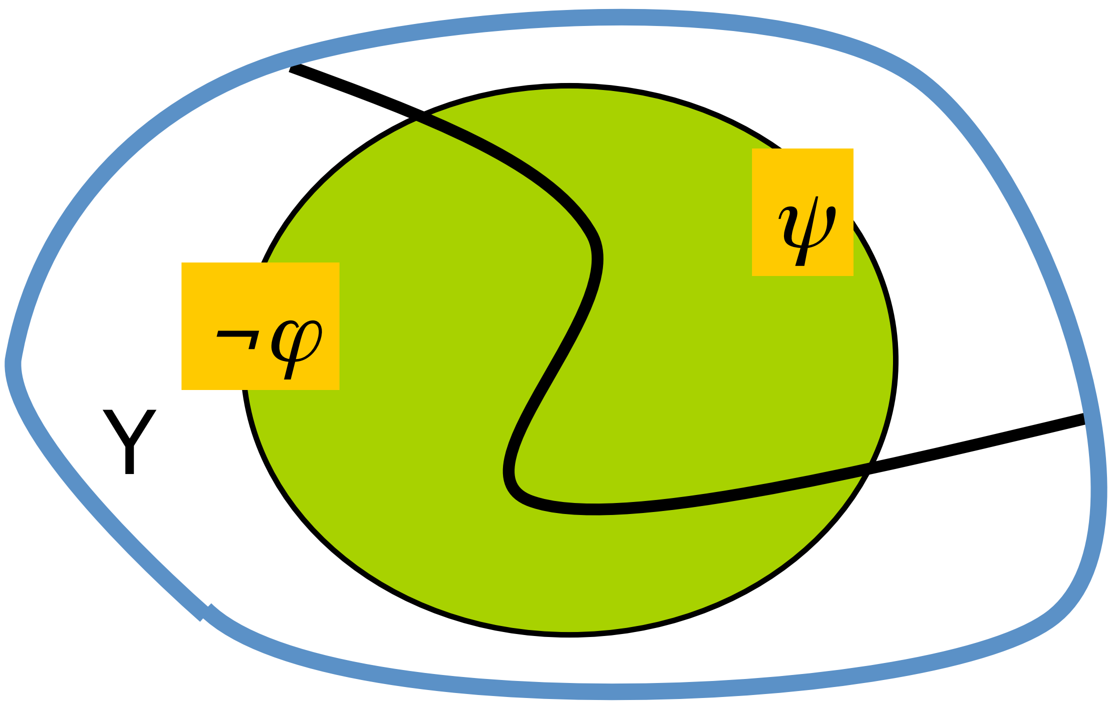
Proof

- If $\varphi \rightarrow \psi$ is valid and x is not free in φ then $\varphi \rightarrow \forall x \psi$ is valid.



Proof

- If $\varphi \rightarrow \psi$ is valid and x is not free in φ then $\varphi \rightarrow \forall x \psi$ is valid.



A special axiom schema

- **Comprehension Axioms:**

$$\forall x(\varphi \vee \neg \varphi),$$

if φ contains no dependence atoms.

Conservative over FO

Corollary 22 *Let ϕ be a first order L-formula of dependence logic. Then:*

1. $\mathcal{M} \models_{\{s\}} \phi$ *if and only if* $\mathcal{M} \models_s \phi$.
2. $\mathcal{M} \models_X \phi$ *if and only if* $\mathcal{M} \models_s \phi$ *for all* $s \in X$.

Examples

Example: even cardinality

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 (=\!(x_2, x_3) \;\wedge\; \neg(x_0 = x_1)$$
$$\wedge\; (x_0 = x_2 \longrightarrow x_1 = x_3)$$
$$\wedge\; (x_1 = x_2 \longrightarrow x_3 = x_0))$$

Example: infinity

$$\exists x_4 \forall x_0 \exists x_1 \forall x_2 \exists x_3 (=(x_2, x_3) \;\wedge\; \neg(x_1 = x_4) \;\wedge\; (x_0 = x_2 \leftrightarrow x_1 = x_3))$$

”There is a bijection to a proper subset.”

Game theoretical semantics

Semantic game of *D*

Beginning of the game

As for first order logic

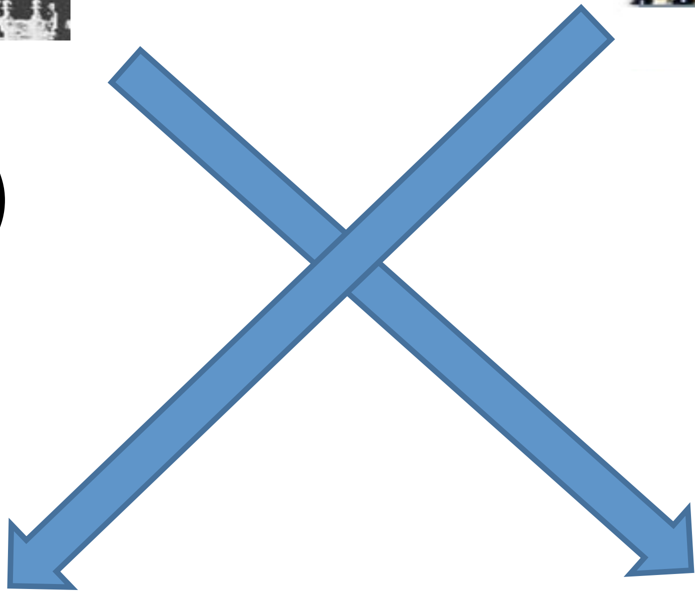
(φ, s)

Conjunction move: “other”

As for first order logic

$(\varphi \wedge \psi, s)$

$(\varphi \wedge \psi, s)$



(φ, s)

(ψ, s)

Disjunction move: “self”

As for first order logic

$(\varphi \vee \psi, s)$

$(\varphi \vee \psi, s)$

(ψ, s)

(φ, s)

Negation move

As for first order logic

$(\neg\varphi, s)$ → (φ, s)

(φ, s) ← $(\neg\varphi, s)$

Existential quantifier move: “me”

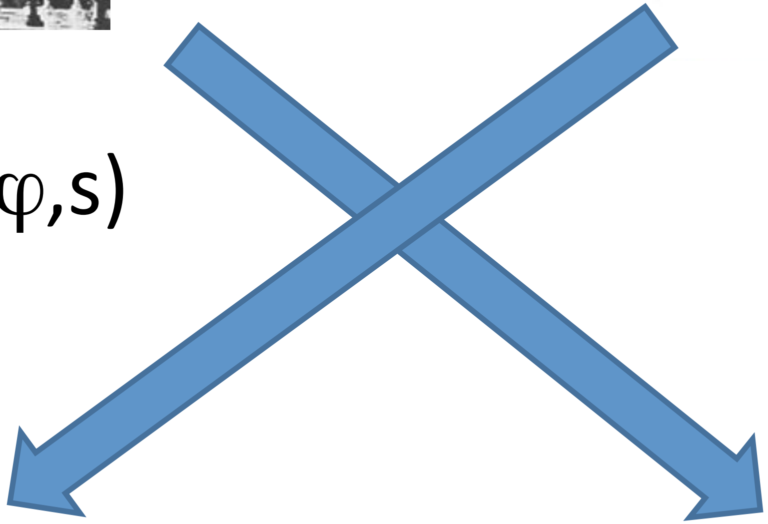
As for first order logic

$(\exists x\varphi, s)$ $(\exists x\varphi, s)$

$(\varphi, s(a/x))$ $(\varphi, s(a/x))$

Universal quantifier move: “other”

As for first order logic



$(\forall x\varphi,s)$ $(\forall x\varphi,s)$

$(\varphi,s(a/x))$ $(\varphi,s(a/x))$

Non-dependence atomic formula

As for first order logic

(φ,s)

false

(φ,s)

false

Dependence atom

New!

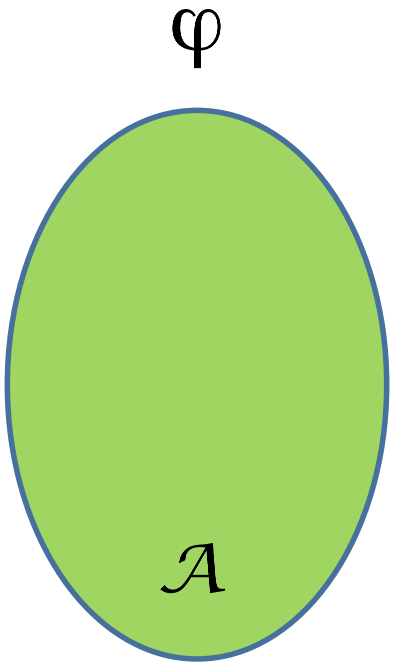
(φ, s)

(φ, s)

Uniform strategy

- A strategy of II is uniform if whenever the game ends in $(=(t_1,\ldots,t_n),s)$ with the same $=(t_1,\ldots,t_n)$ and the same values of $t_1,\ldots,t_{n-1}$, then also the value of t_n is the same.
- Imperfect information game!

Game theoretical semantics of *D*

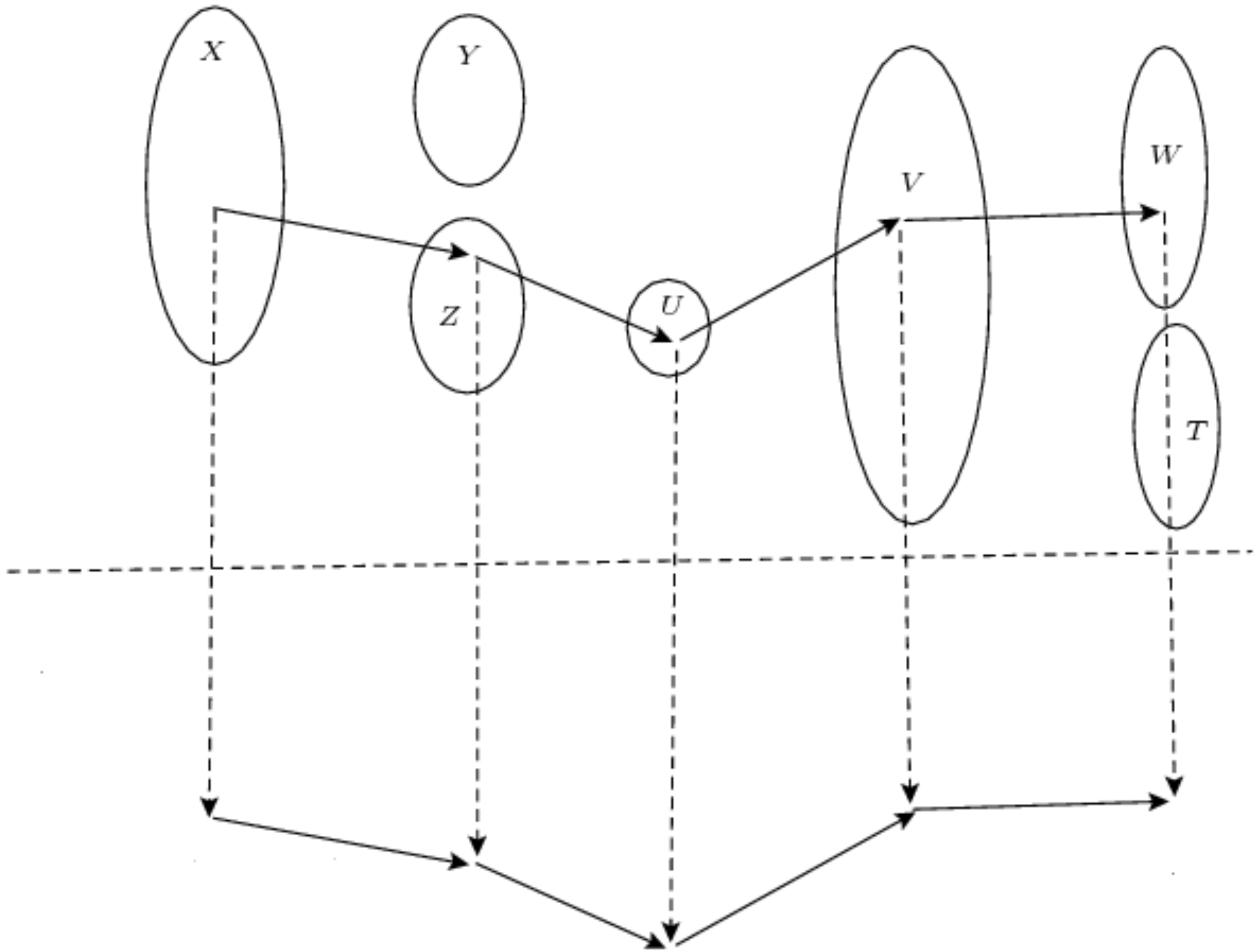


φ is true in 𝒜 if and only if II has a uniform winning strategy

The power-strategy

- **Winning strategy of II:** keep holding an auxiliary team X and make sure that if you hold a pair (φ,s), then s$\in$X and X is of type φ, and if he holds (φ,s), then s$\in$X and X is of type $\neg\varphi$.

- **This is uniform:** Suppose game is played twice and it ends first in $(=(t_1,\ldots,t_n),s)$ and then in $(=(t_1,\ldots,t_n),s')$. In both cases II held a team. W.l.o.g. the team is both times the same team X. Now s,s' $\in$X and X is of type $=(t_1,\ldots,t_n)$. So if the values of $t_1,\ldots,t_{n-1}$ are the same, then also the value of t_n is the same.


X
Y
Z
U
V
W
T

The right team from winning

- Suppose II has a uniform winning strategy τ starting from $(\varphi,\emptyset)$.
- **Idea**: Let X_ψ be the set of assignments s such that (ψ,s) is a position in the game, II playing τ.
- **By induction on ψ**: If II holds (ψ,s), then X_ψ is of type ψ. If I holds (ψ,s), then X_ψ is of type $\neg\psi$.

Model theory of dependence logic

- Basic reduction:

$\phi(x_{i_1}, ..., x_{i_n})$

$d \in \{0, 1\}$

⇒

Σ^1_1-*sentence* $\tau_{d,\phi}(S)$

New predicate

Equivalent

1. $(\phi, X, d) \in \mathcal{T}$
2. $(\mathcal{M}, X) \models \tau_{d,\phi}(S)$

Team becomes predicate

φ is $=(t_1(x_{i_1}, \ldots, x_{i_n}), \ldots, t_m(x_{i_1}, \ldots, x_{i_n}))$

$\tau_{1,\phi}(S)$ =

$$\begin{aligned}\forall x_{i_1} \ldots \forall x_{i_n} \forall x_{i_n+1} \ldots \forall x_{i_n+n}((&Sx_{i_1} \ldots x_{i_n} \wedge Sx_{i_n+1} \ldots x_{i_n+n} \wedge \\ &t_1(x_{i_1}, \ldots, x_{i_n}) = t_1(x_{i_n+1}, \ldots, x_{i_n+n}) \wedge \\ &\ldots \\ &t_{m-1}(x_{i_1}, \ldots, x_{i_n}) = t_{m-1}(x_{i_n+1}, \ldots, x_{i_n+n})) \\ &\rightarrow t_m(x_{i_1}, \ldots, x_{i_n}) = t_m(x_{i_n+1}, \ldots, x_{i_n+n}))\end{aligned}$$

$\tau_{0,\phi}(S)$ =

$$\forall x_{i_1} \ldots \forall x_{i_n} \neg S x_{i_1} \ldots x_{i_n}$$

φ is $(\psi(x_{j_1}, ..., x_{j_p}) \vee \theta(x_{k_1}, ..., x_{k_q}))$

$\tau_{1,\phi}(S)$ =

$$\exists R \exists T(\tau_{1,\psi}(R) \wedge \tau_{1,\theta}(T) \wedge \\ \forall x_{i_1} ... \forall x_{i_n}(S x_{i_1} ... x_{i_n} \rightarrow (R x_{j_1} ... x_{j_p} \vee T x_{k_1} ... x_{k_q})))$$

$\tau_{0,\phi}(S)$ =

$$\exists R \exists T(\tau_{0,\psi}(R) \wedge \tau_{0,\theta}(T) \wedge \\ \forall x_{i_1} ... \forall x_{i_n}(S x_{i_1} ... x_{i_n} \rightarrow (R x_{j_1} ... x_{j_p} \wedge T x_{k_1} ... x_{k_q})))$$

Negation

ϕ is $\neg\psi$. $\tau_{d,\phi}(S)$ is the formula $\tau_{1-d,\psi}(S)$.

Existential quantifier

Suppose $\phi(x_{i_1}, ..., x_{i_n})$ is the formula $\exists x_{i_{n+1}} \psi(x_{i_1}, ..., x_{i_{n+1}})$.
$\tau_{1,\phi}(S)$ is the formula

$$\exists R(\tau_{1,\psi}(R) \wedge \forall x_{i_1} ... \forall x_{i_n}(S x_{i_1} ... x_{i_n} \rightarrow \exists x_{i_{n+1}} R x_{i_1} ... x_{i_{n+1}}))$$

and $\tau_{0,\phi}(S)$ is the formula

$$\exists R(\tau_{0,\psi}(R) \wedge \forall x_{i_1} ... \forall x_{i_n}(S x_{i_1} ... x_{i_n} \rightarrow \forall x_{i_{n+1}} R x_{i_1} ... x_{i_{n+1}})).$$

Corollary

Both are ESO!

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{M} \models \tau_{1,\phi}.$$

$$\mathcal{M} \models \neg\phi \text{ if and only if } \mathcal{M} \models \tau_{0,\phi}.$$

Application

Theorem 58 (Compactness Theorem of $\mathcal{D}$) *Suppose* Γ *is an arbitrary set of sentences of dependence logic such that every finite subset of* Γ *has a model. Then* Γ *itself has a model.*

Application

In countable vocabulary

Theorem 59 (Löwenheim-Skolem Theorem of $\mathcal{D}$) *Suppose ϕ is a sentence of dependence logic such that ϕ either has an infinite model or has arbitrarily large finite models. Then ϕ has models of all infinite cardinalities, in particular, ϕ has a countable model and an uncountable model.*

Application

Theorem 61 (Separation Theorem) *Suppose ϕ and ψ are sentences of dependence logic such that ϕ and ψ have no models in common. Let the vocabulary of ϕ be L and the vocabulary of ψ be L'. Then there is a sentence θ of $\mathcal{D}$ in the vocabulary $L \cap L'$ such that every model of ϕ is a model of θ, but θ and ψ have no models in common. In fact, θ can be chosen to be first order.*

Application

Theorem 62 (Failure of the Law of Excluded Middle) *Suppose ϕ and ψ are sentences of dependence logic such that for all models $\mathcal{M}$ we have $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \not\models \psi$. Then ϕ is logically equivalent to a first order sentence θ such that ψ is logically equivalent to $\neg\theta$.*

Non-determinacy

Definition 63 *A sentence ϕ of dependence logic is called* determined in $\mathcal{M}$ *if* $\mathcal{M} \models \phi$ *or* $\mathcal{M} \models \neg\phi$. *Otherwise ϕ is called* non-determined in $\mathcal{M}$. *We say that ϕ is* determined *if ϕ is determined in every structure.*

Corollary 64 *Every determined sentence of dependence logic is strongly logically equivalent to a first order sentence.*

Skolem Normal Form

Theorem 66 (Skolem Normal Form Theorem) *Every Σ_1^1 formula ϕ is logically equivalent to an existential second order formula*

$$\exists f_1 \dots \exists f_n \forall x_1 \dots \forall x_m \psi, \tag{4.1}$$

where ψ is quantifier free and $f_1, \dots, f_n$ are function symbols. The formula (4.1) is called a Skolem Normal Form of ϕ.

From ESO to D

Theorem 68 ([4],[30]) *For every* Σ^1_1*-sentence* ϕ *there is a sentence* ϕ^* *in dependence logic such that for all* $\mathcal{M}$*:* $\mathcal{M} \models \phi \iff \mathcal{M} \models \phi^*$.

Sketch of proof

$$\exists f \forall x \forall y \phi(x, y, f(x, y), f(y, x))$$

$$\begin{aligned}\forall x \forall y \exists z \forall x' \forall y' \exists z' \, (&=\!(x', y', z') \wedge \\ &((x = x' \wedge y = y') \to z = z') \wedge \\ &((x = y' \wedge x' = y) \to \phi(x, y, z, z'))\end{aligned}$$

Current developments

- Also independence atoms.
- See Doctoral Thesis of Pietro Galliani: www.illc.uva.nl/Research/Dissertations/DS-2012-07.text.pdf
- See paper by Kontinen-Väänänen: http://arxiv.org/abs/1208.0176
- See paper by Grädel-Väänänen: http://logic.helsinki.fi/people/jouko.vaananen/graedel_vaananen.pdf