Lecture 4

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Model existence game

Questions we can address

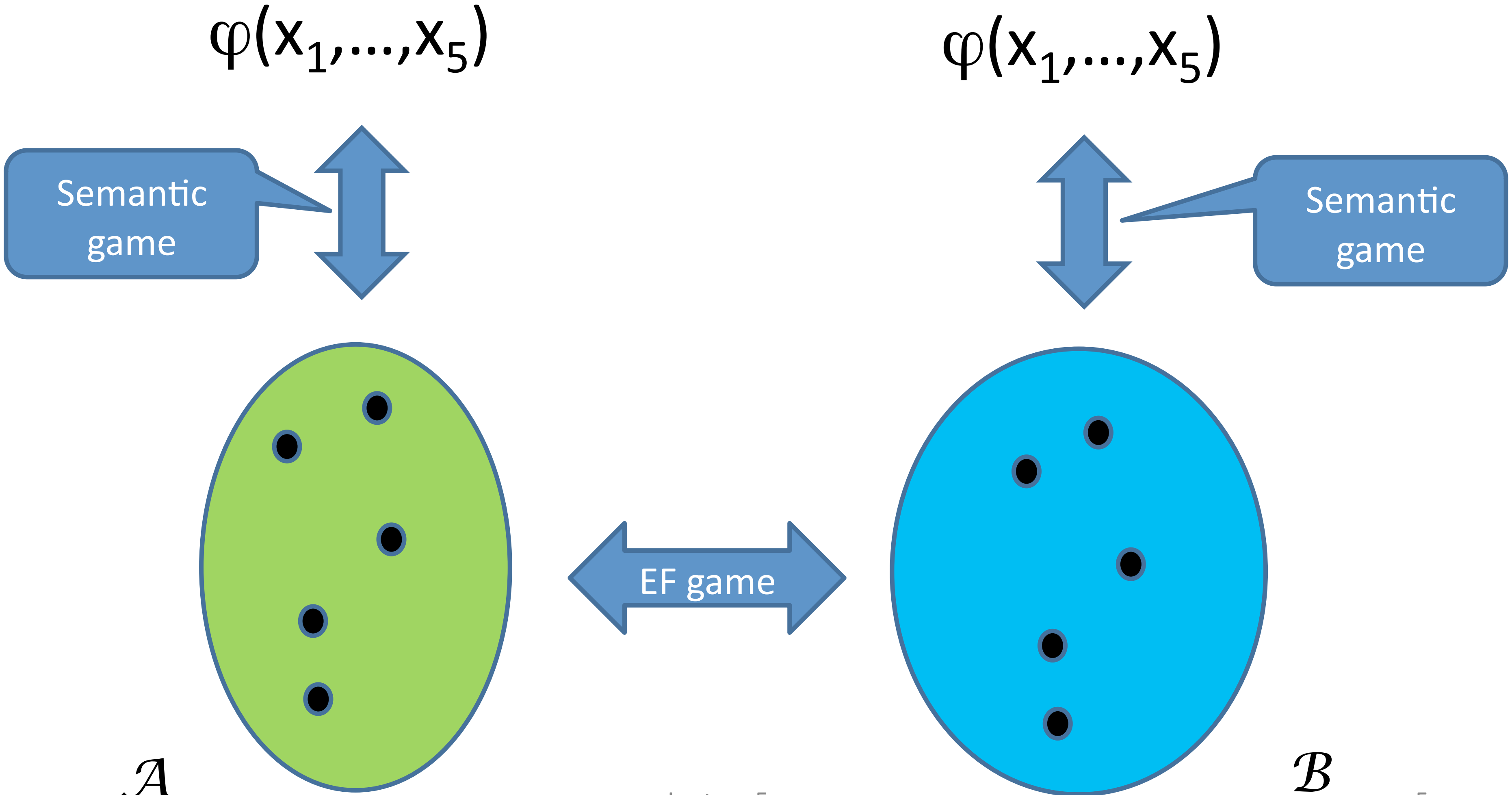
- Does φ have a model?
- Does T have a model in which φ holds?
- Is φ true in every model of T?
- Is φ true in every model?

Model Existence Game

- MEG(T,L), where T is an L-theory.
- Player I (``Max”) claims T has no models, rather it is contradictory.
- Player II (``Susan”) claims T has a model, and she knows one (but she can bluff).
- Player II tries to play only sentences supposedly true in the supposed model.
- Player I tries to challenge this.

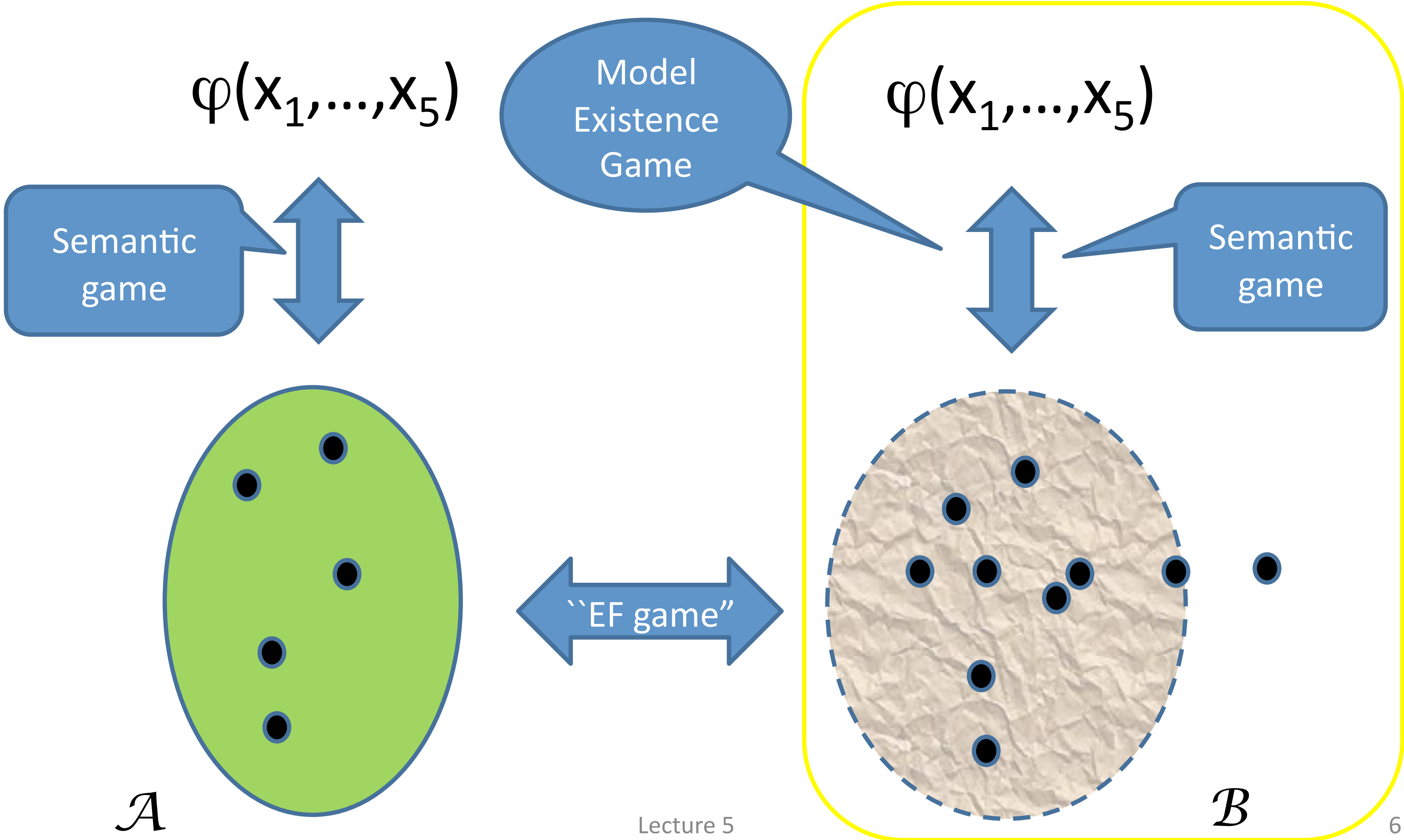
Asynchronous parallel games

(Strategic Balance of Logic)



Model existence game

(Strategic Balance of Logic)

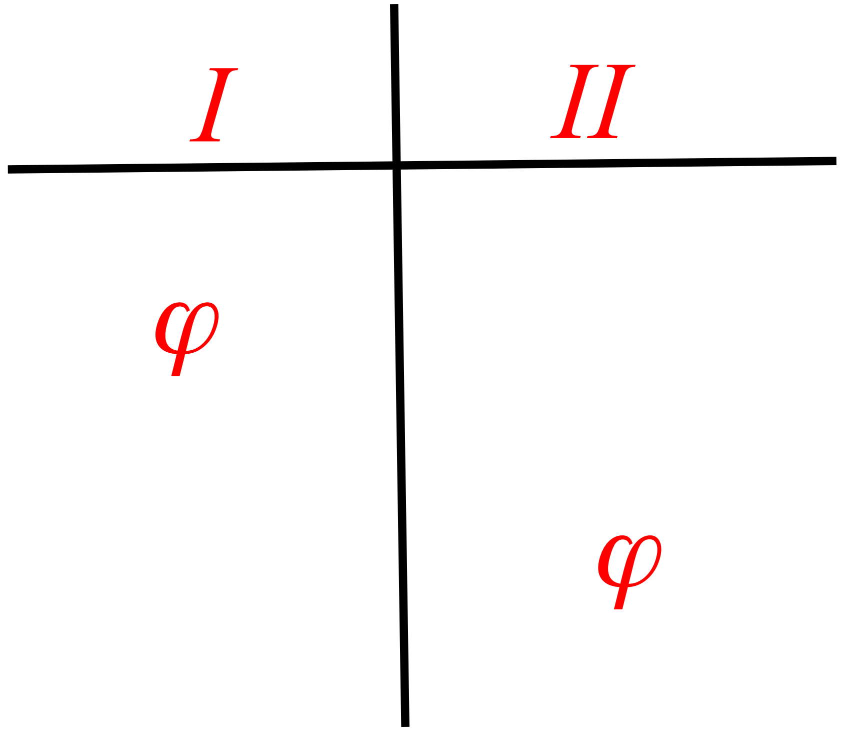


Rules of the game

- C = a new countably infinite set of constants ("elements of the model that II knows")
- The moves are L∪C-sentences in NNF
  - Negation Normal Form
    - $\wedge$ $\vee$ $\neg$ $\forall$ $\exists$
    - negation only in front of atomic sentences

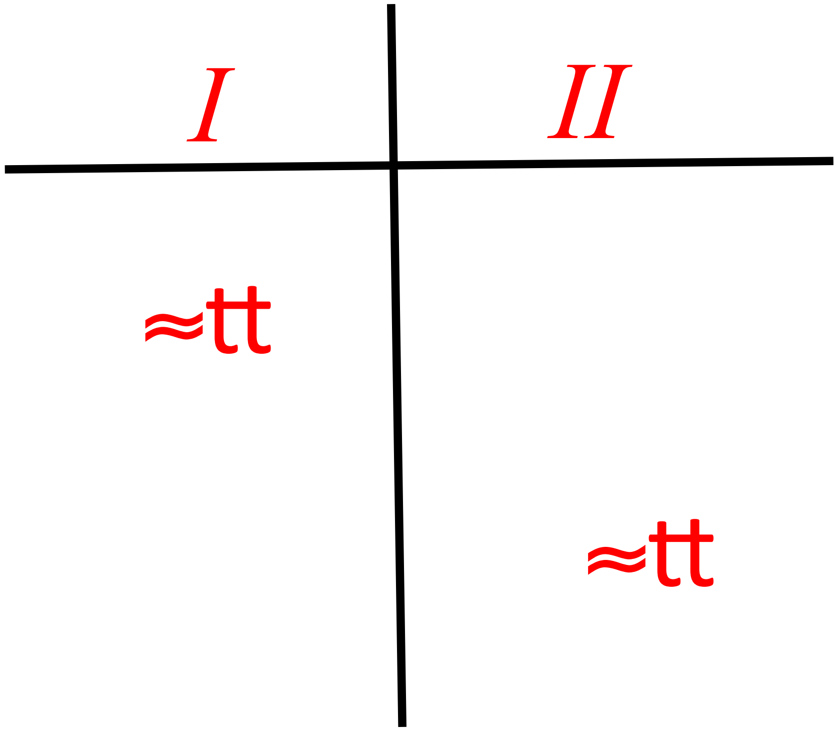
Theory move

- I plays some φ∈T.
- II accepts.



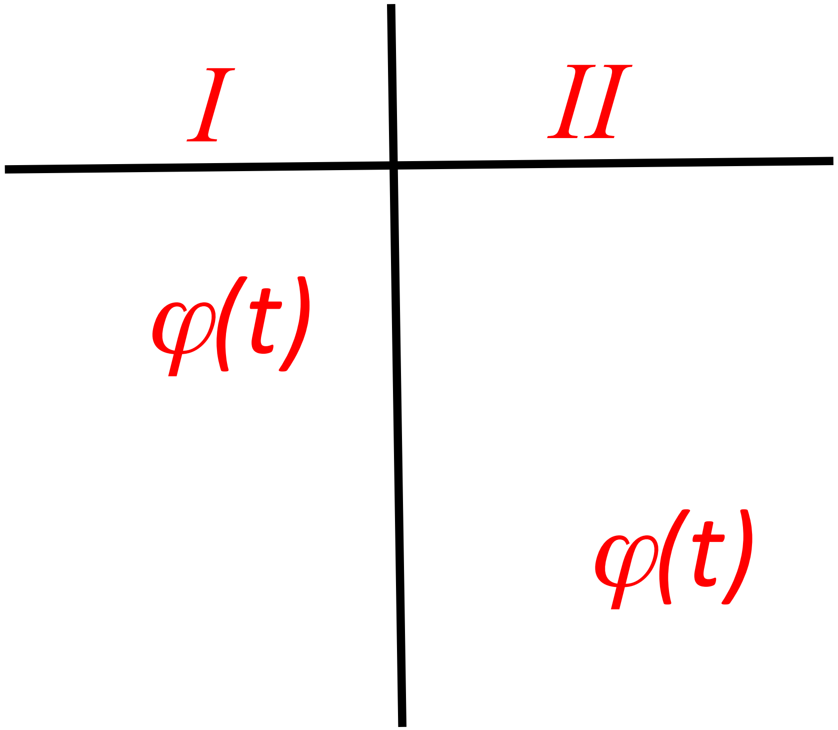
Equation move

- I plays some ≈tt.
- II accepts.



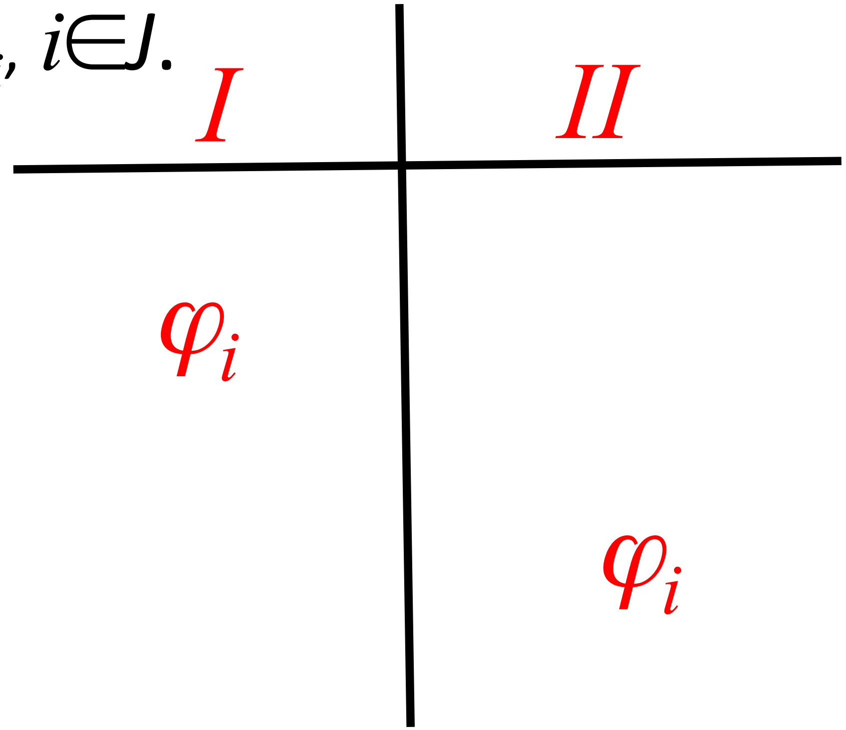
Substitution move

- I picks ≈ct and φ(c), previously played by II, and plays φ(t).
- II accepts.



Conjunction move

- I picks some $\bigwedge_{i\in J}\varphi_i$, previously played by II, and some φ_i, $i\in J$.
- II accepts.



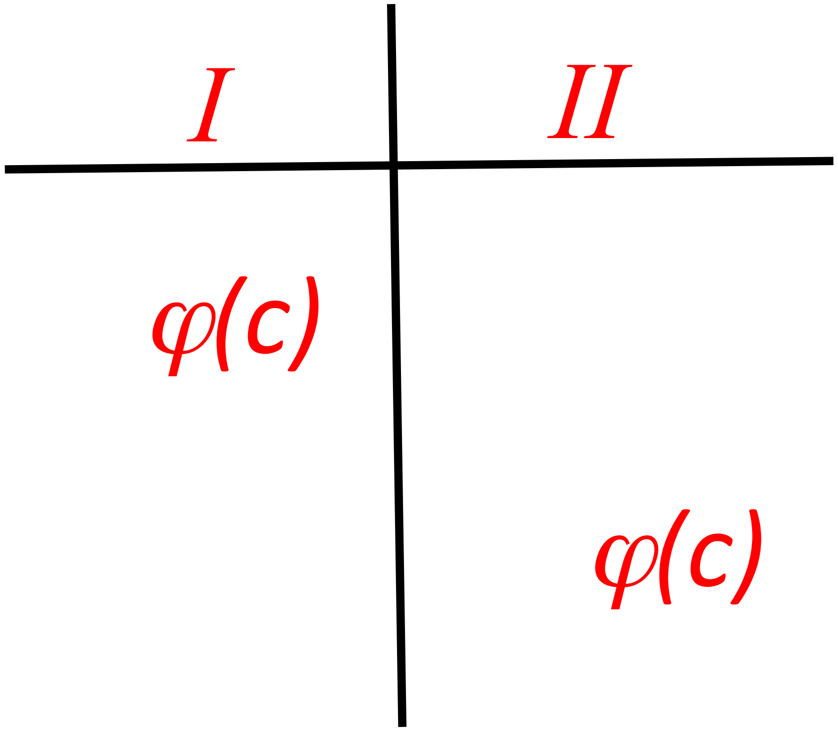
Disjunction move

- I plays some $\bigvee_{i\in J}\varphi_i$, previously played by II.
- II chooses φ_i, $i\in I$.

I	*II*
$\bigvee_{i\in J}\varphi_i$	
	φ_i

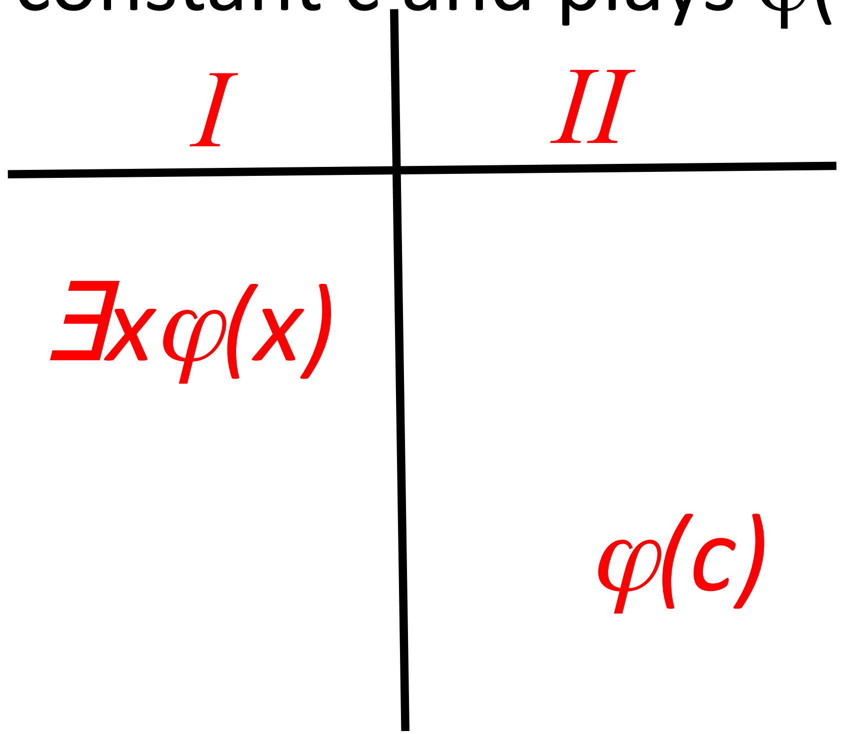
Universal quantifier move

- I picks some $\forall x \varphi(x)$ that II has previously played, and a constant c. Then he plays $\varphi(c)$.
- II accepts.



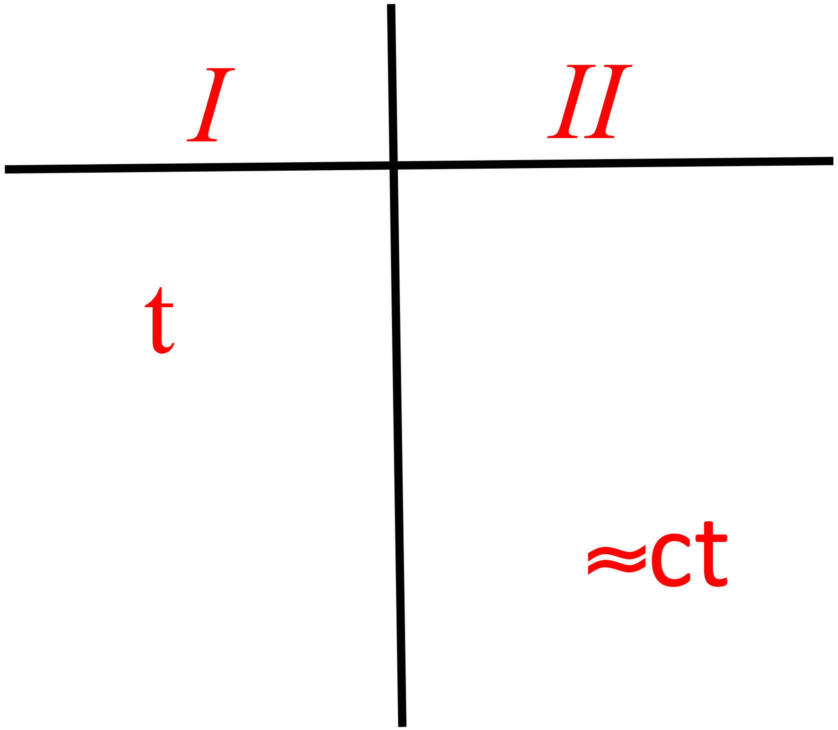
Existential quantifier move

- I plays some $\exists x\varphi(x)$ that II has previously played.
- II chooses a constant c and plays φ(c).



Constant move

- I plays some t.
- II plays ≈ct for some constant c.

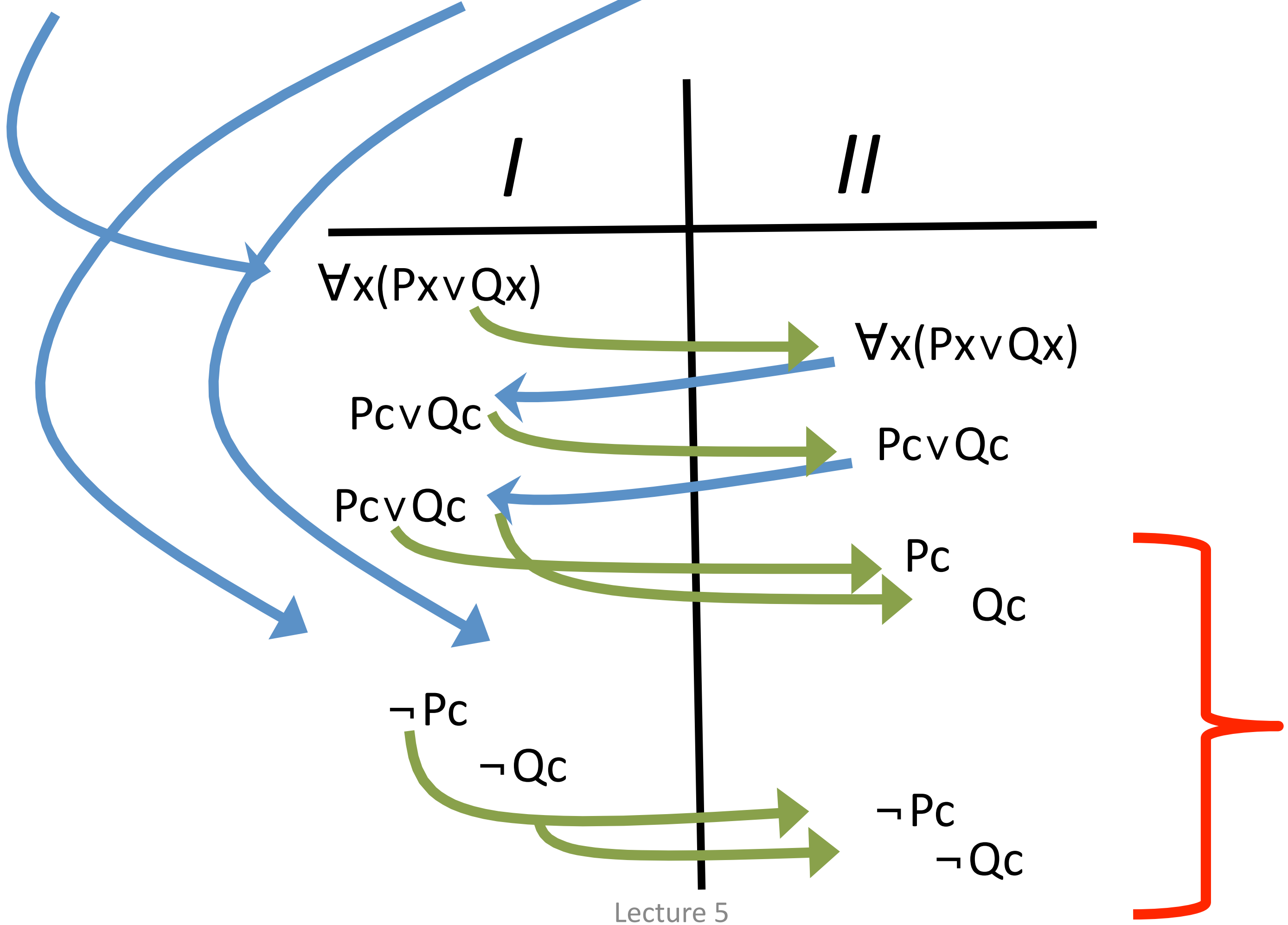


I wins if ...

- For some atomic sentence both φ and ¬φ are played by II.
- Otherwise II wins.
- Closed game.
- Determined.

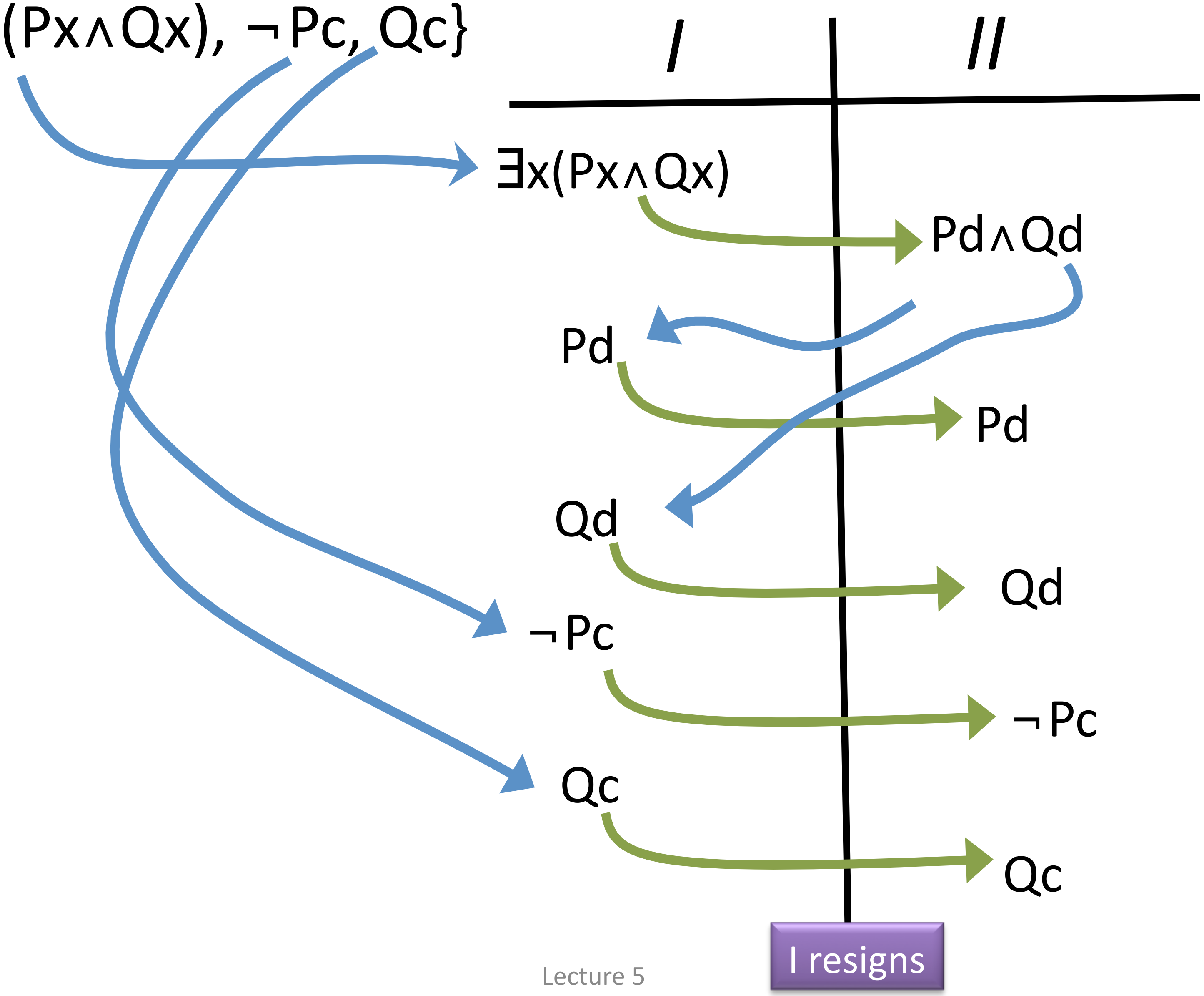
Example where I wins

$T=\{\forall x(Px \vee Qx), \neg Pc, \neg Qc\}$



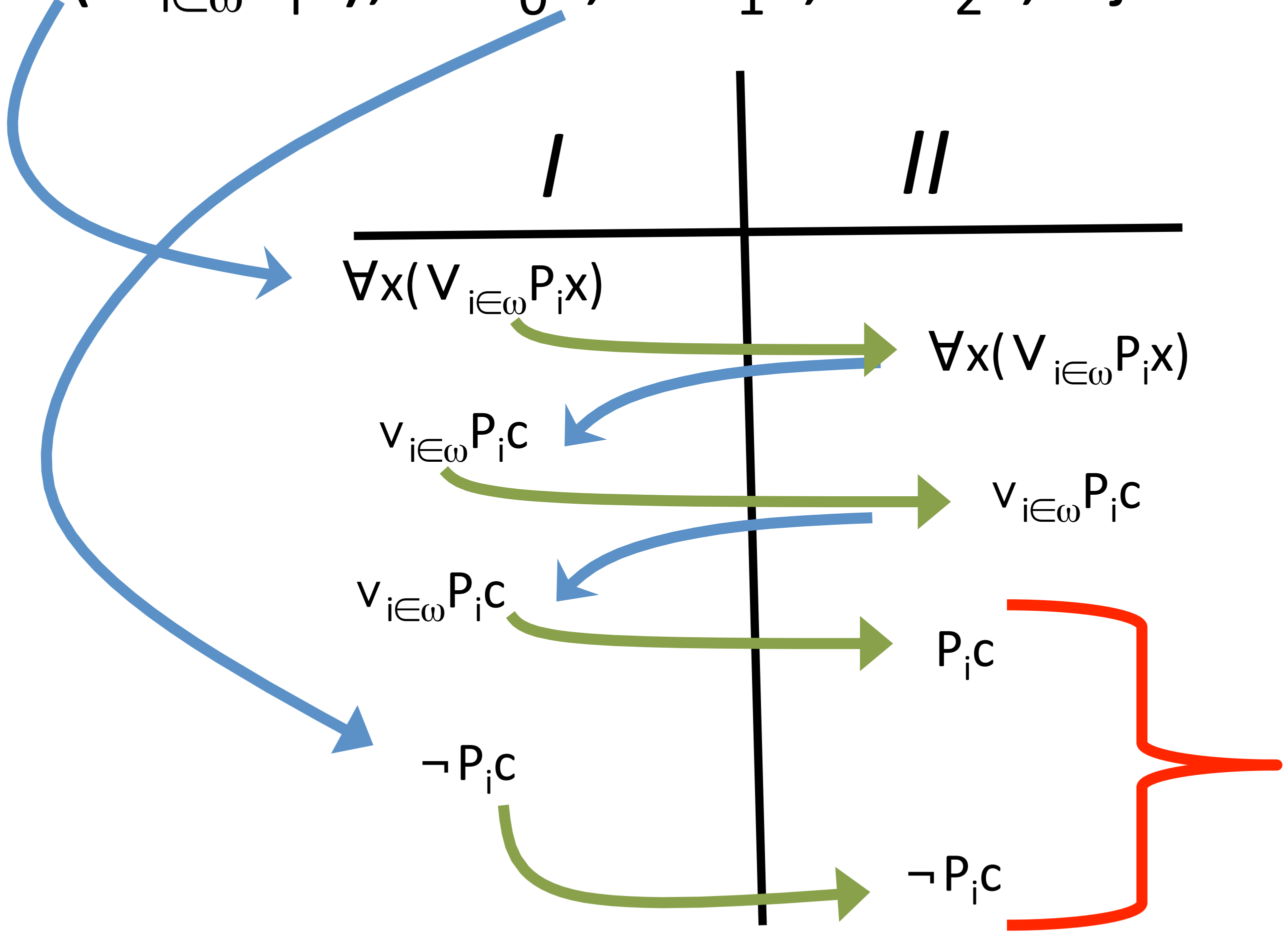
Example where II wins

$T=\{\exists x(Px \wedge Qx), \neg Pc, Qc\}$



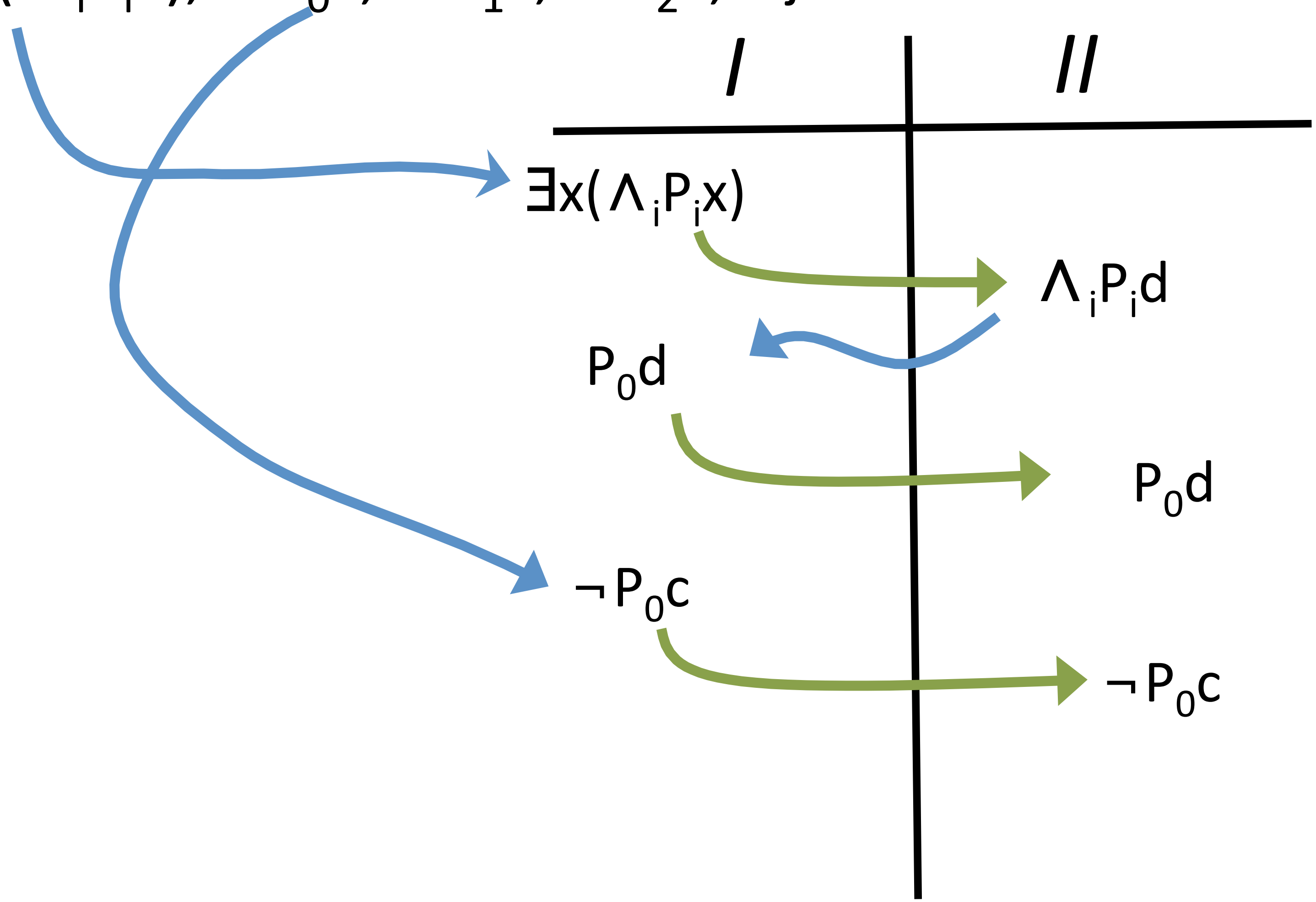
Example where I wins

$T=\{\forall x(\vee_{i\in\omega}P_ix), \neg P_0c, \neg P_1c, \neg P_2c,\ldots\}$



Example where II wins

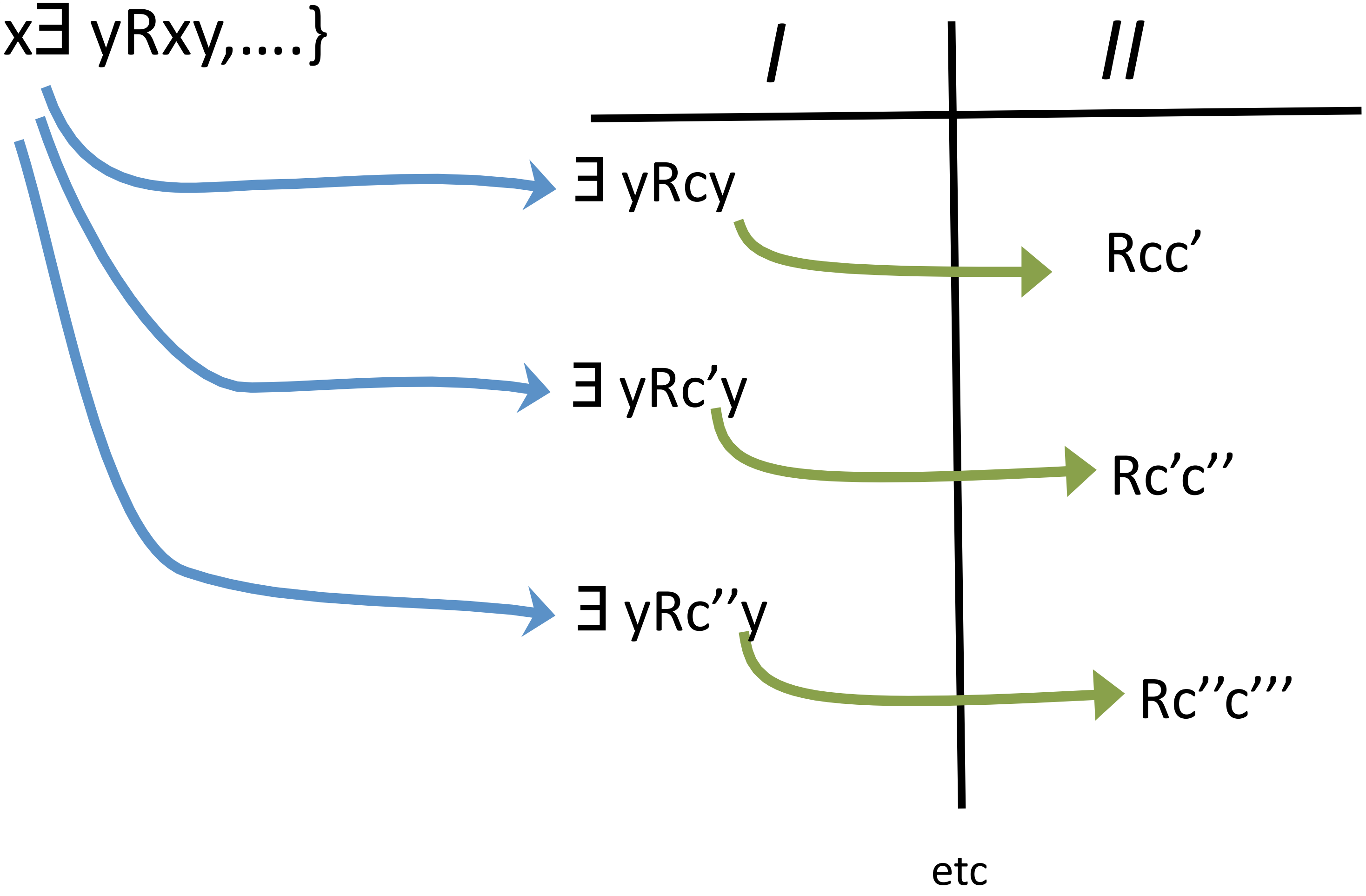
$T=\{\exists x(\bigwedge_i P_i x), \neg P_0 c, \neg P_1 c, \neg P_2 c, \ldots\}$



I resigns

Example where the game is infinite

$T=\{\forall x \exists\, yRxy, \ldots.\}$



Basic Theorem

- Susan has a winning strategy in the model existence game on T if and only if T has a model.

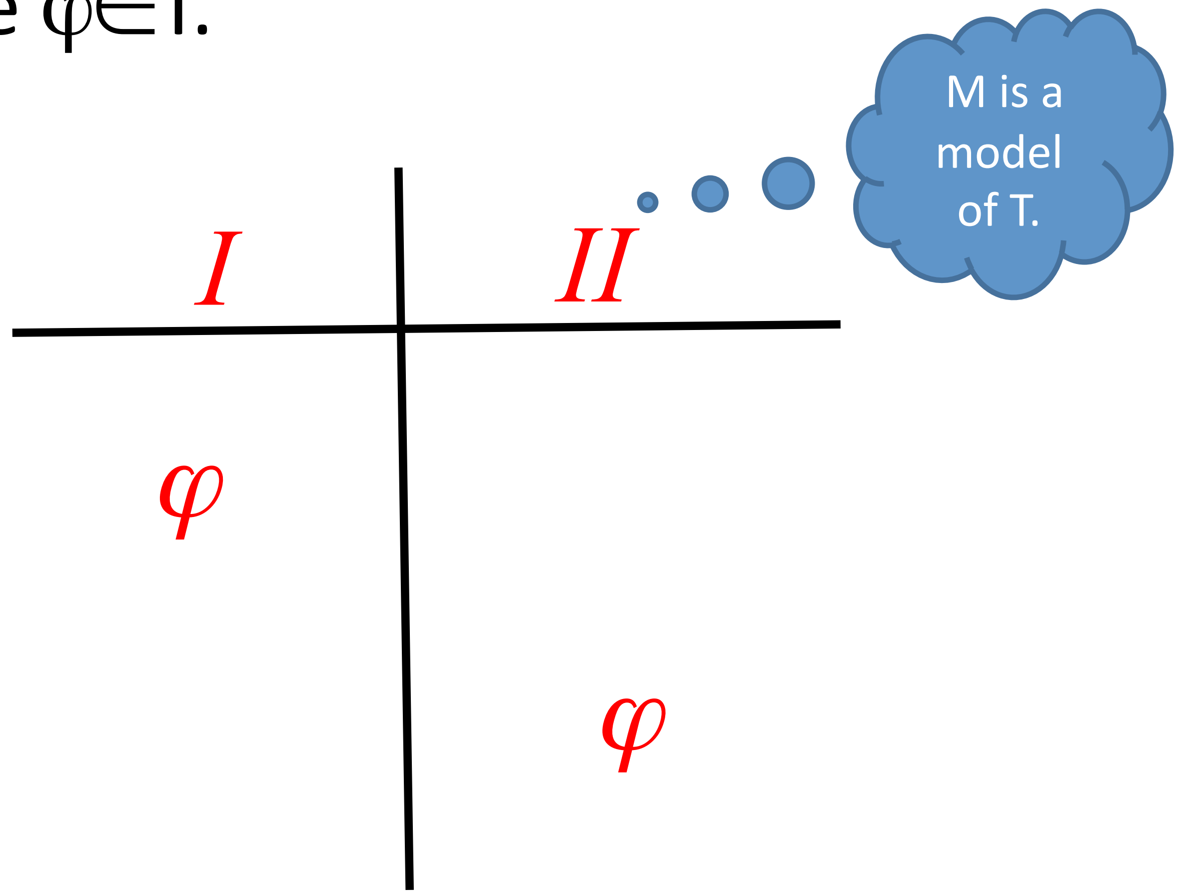
A model strategy

Suppose T has a model M.

During the game II interprets the constants of C in M so that all sentences that she has played are true in M.

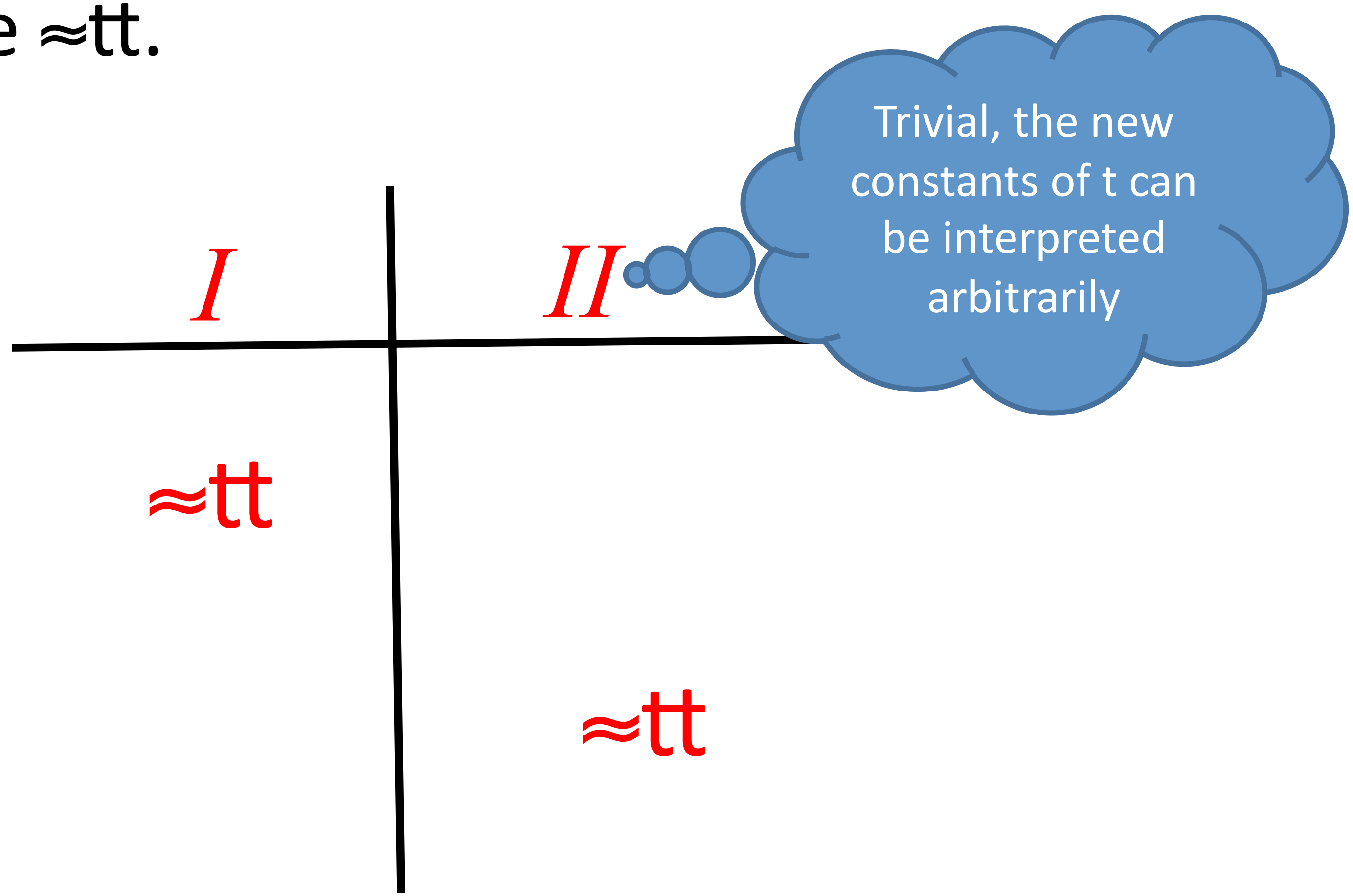
Theory move preserves truth

- I plays some φ∈T.
- II accepts.



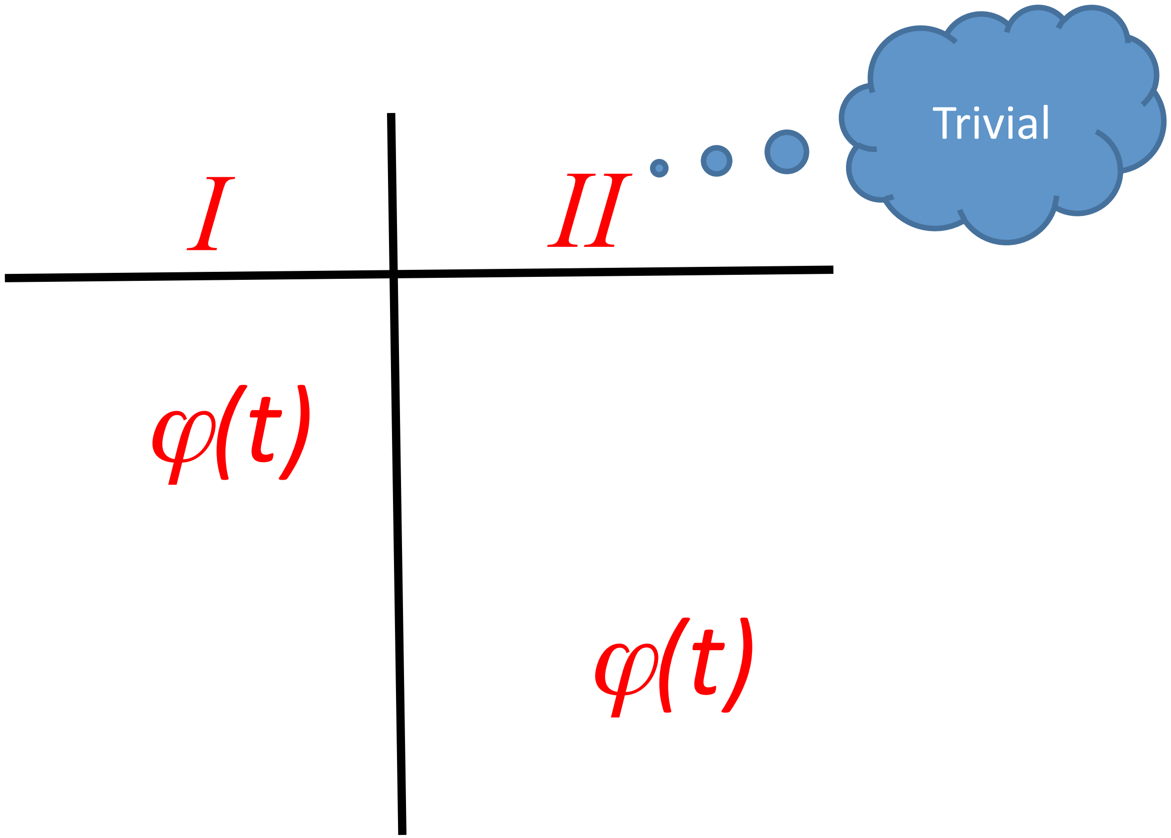
Equation move preserves truth

- I plays some ≈tt.
- II accepts.



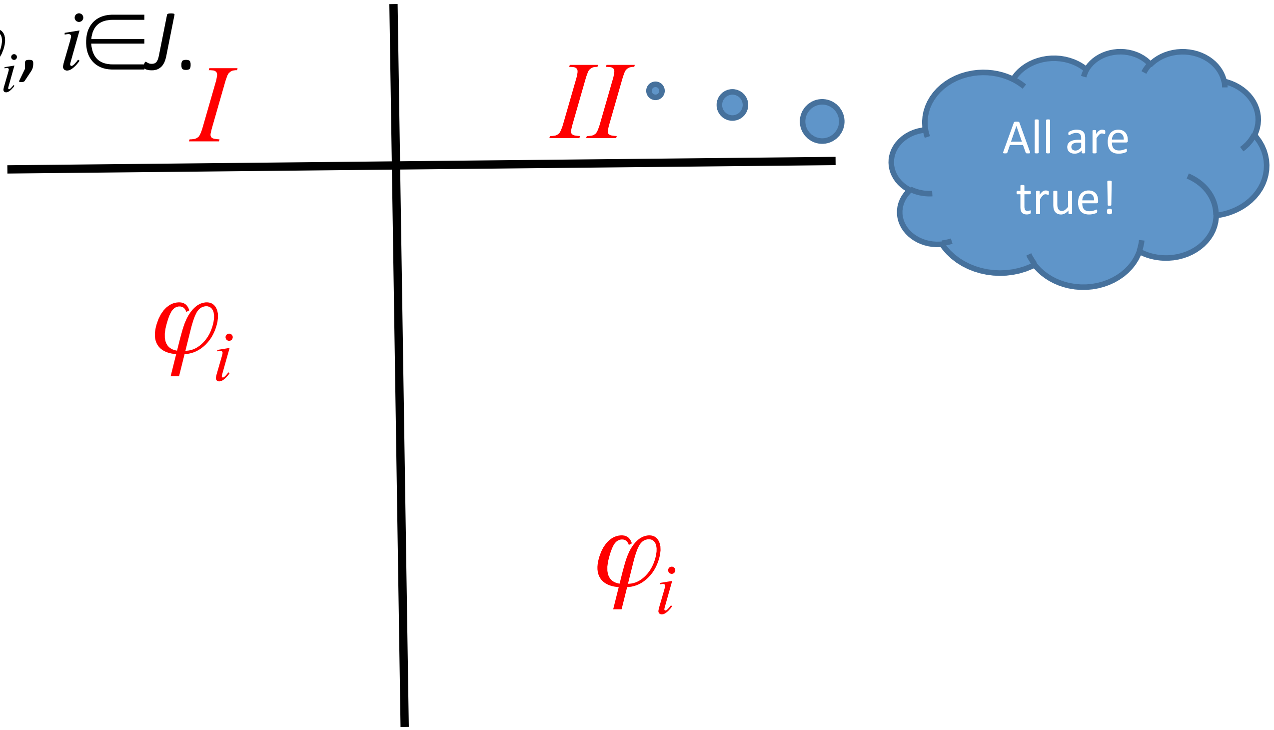
Substitution move preserves truth

- I picks previously played ≈ct and φ(c), and plays φ(t).
- II accepts.



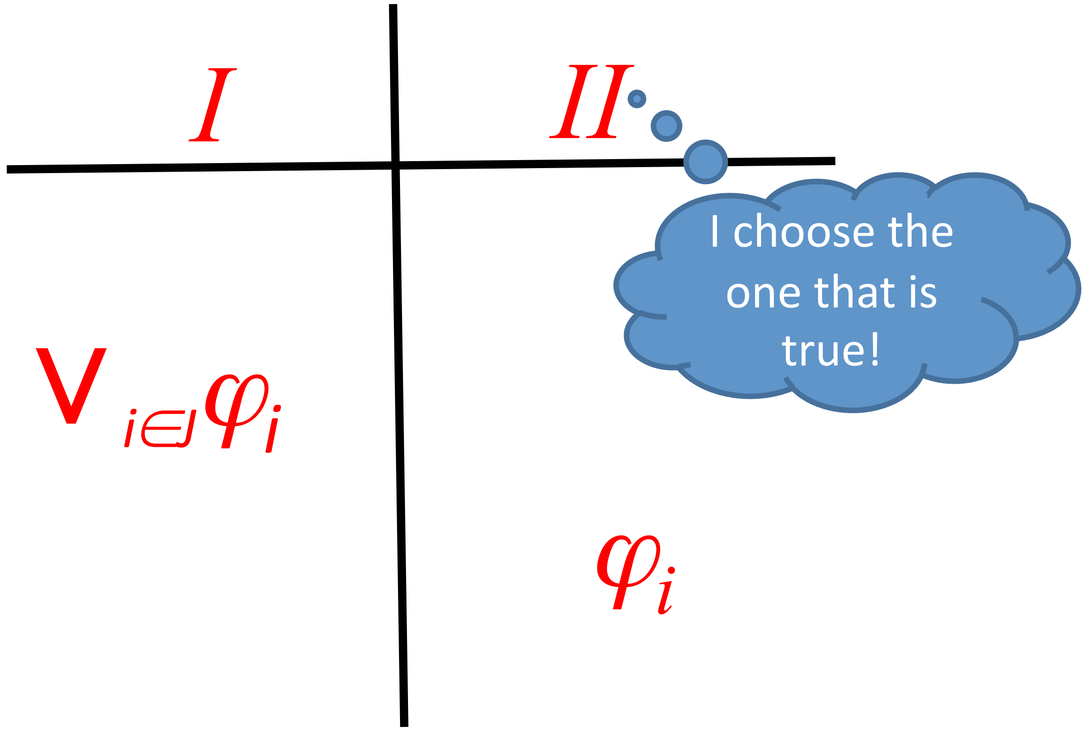
Conjunction move

- I plays some $\bigwedge_{i \in J} \varphi_i$, previously played by II, and some φ_i, $i \in J$.
- II accepts.



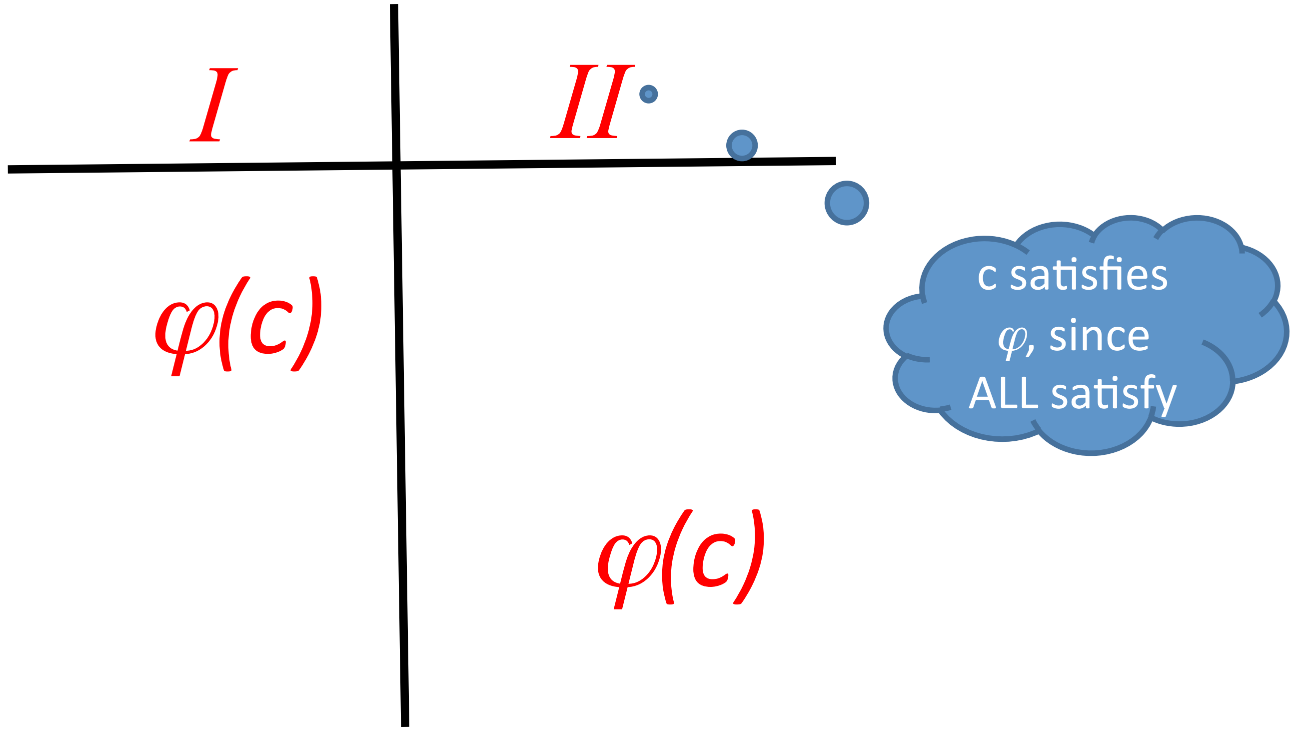
Disjunction move

- I plays some $\bigvee_{i\in J}\varphi_i$, previously played by II.
- II chooses φ_i, $i\in J$.



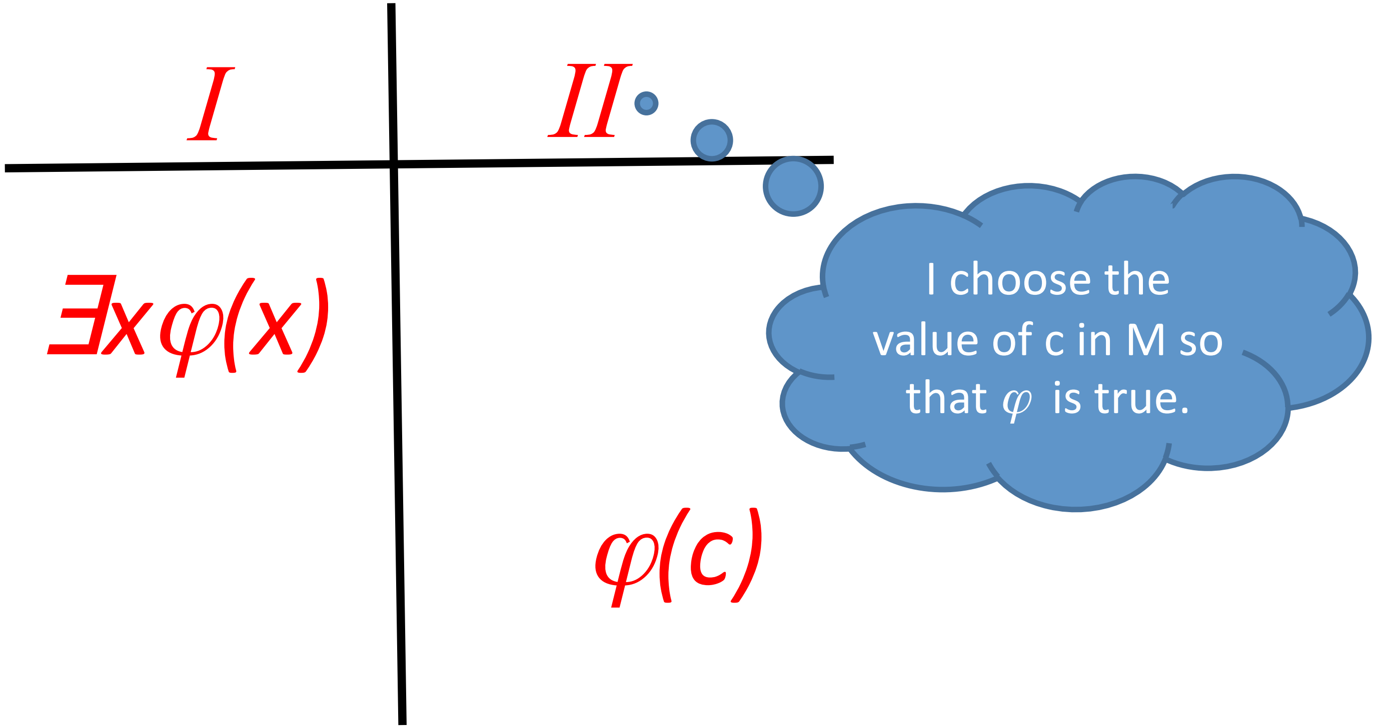
Universal quantifier move

- I picks some ∀xφ(x), previously played by II, and a constant c. Then he plays φ(c).
- II accepts.



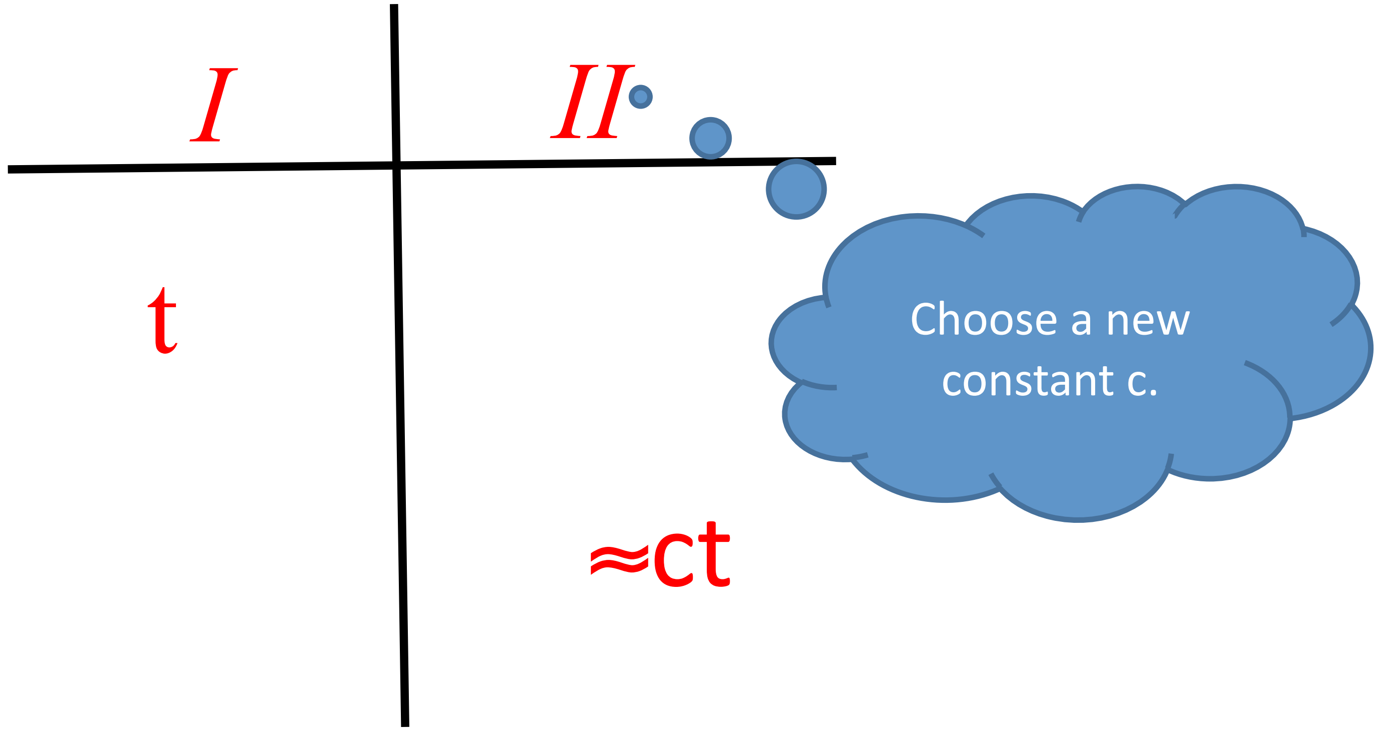
Existential quantifier move

- I plays some $\exists x\varphi(x)$, previously played by II.
- II chooses a constant c and plays φ(c).



Constant move

- I plays some t.
- II plays ≈ct for some constant c.



II wins

- II wins because it cannot happen that for some (atomic) φ both φ and $\neg\varphi$ are true in M.

An enumeration strategy

Suppose II has a winning strategy.

There is a strategy of I which enumerates all possibilities.

It turns out that since II wins even against the enumeration strategy, the theory T has to have a model.

The idea

I plays every $\varphi \in T$ as a **theory move**.

I plays every possible equation $\approx tt$ as an **equation move**.

If $\approx ct$ and $\varphi(c)$ have been played, then I plays $\varphi(t)$ as a **substitution move**.

If $\wedge \varphi_i$ has been played, then each φ_i is played in **conjunction moves**.

If $\vee \varphi_i$ has been played by II, then also I plays it as a **disjunction move**.

If $\forall x \varphi(x)$ has been played, and a c is a constant , then I plays $\varphi(c)$ as a **universal quantifier move**.

If $\exists x \varphi(x)$ has been played by II, I plays it as an **existential quantifier move**.

Player I plays every term t as a **constant move**.

In detail

$$T = \{\phi_n : n \in \mathbb{N}\}$$
$$C = \{c_n : n \in \mathbb{N}\}$$
$$Trm = \{t_n : n \in \mathbb{N}\}$$

1. If $n = 0$, then $x_n = \varphi$.
2. If $n = 2 \cdot 3^i$, then x_n is $\approx c_i c_i$.
3. If $n = 4 \cdot 3^i \cdot 5^j \cdot 7^k \cdot 11^l$, y_i is $\approx c_j t_k$, and y_l is $\varphi(c_j)$, then x_n is $\varphi(c_i)$.
4. If $n = 8 \cdot 3^i \cdot 5^j$ and y_i is $\bigwedge_{m \in \mathbb{N}} \varphi_m$, then x_n is φ_j.
5. If $n = 16\ 3^i$ and y_i is $\bigvee_{m \in \mathbb{N}} \varphi_m$, then x_n is $\bigvee_{m \in \mathbb{N}} \varphi_m$.
6. If $n = 32 \cdot 3^i \cdot 5^j$, y_i is $\forall x \phi(x)$, then x_n is $\phi(c_j)$.
7. If $n = 64 \cdot 3^i$, and y_i is $\exists x \phi(x)$, then x_n is $\exists x \phi(x)$.
8. If $n = 128 \cdot 3^i$, then x_n is t_i.

Constructing the model

- Let H be all the responses of II.
- Define $c \sim d$ if $\approx cd$ is in H.
- Equivalence relation, even congruence.
- $M=\{[c]:c \in C\}$
- $R^M[c_1]\ldots[c_n]$ iff $Rc_1\ldots c_n \in H$.
- $f^M[c_1]\ldots[c_n]=[d]$ iff $\approx dfc_1\ldots c_n \in H$.

An easy induction

$$\varphi(c_1,\ldots,c_n) \in H \rightarrow M \models \varphi(c_1,\ldots,c_n)$$

I has played every sentence of T.
Hence T is contained in H.
Hence M is a model of T.

Susan wins the semantic game on M and T

Susan makes sure that if she plays $\varphi(c_1,\ldots,c_n)$, then

$\varphi(c_1,\ldots,c_n) \in H$

and if Max plays $\varphi(c_1,\ldots,c_n)$, then

$\neg\varphi(c_1,\ldots,c_n) \in H$

Hence M is a model of T.

Special feature of M

- Every element of M is the interpretation of a constant symbol from C.
- M is countable.

Game-theoretic proofs

φ is true in every model of T

if and only if

Player I has a winning strategy in MEG($T \cup \{\neg\varphi\}$,L).

Compactness strategy

- **Compactness Theorem: Suppose every finite subset of T has a model. Then T has a model.**

- **Strategy of II: Play so that**

  T ∪{sentences you have played}

  is finitely consistent.
- **This is possible if T has only finite conjunctions and disjunctions!**

Interpolation theorem in $L_{\omega_1\omega}$

If $\varphi(P,R)$ and $\psi(P,S)$ are given and

$$\models \varphi(P,R) \rightarrow \psi(P,S),$$

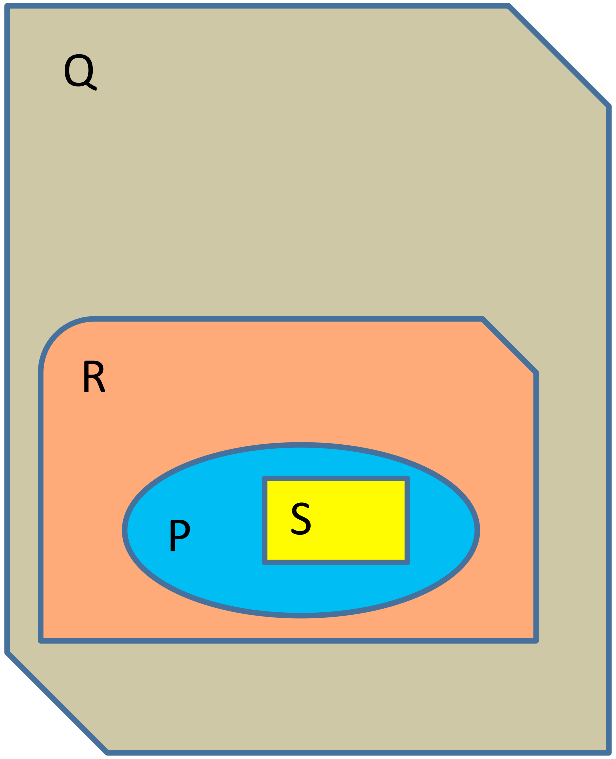
then there is a $\theta(P)$ such that

$$\models \varphi(P,R) \rightarrow \theta(P)$$

and

$$\models \theta(P) \rightarrow \psi(P,S).$$

Example



$$\phi = \forall x(Px \to Rx) \land \forall x(Rx \to Qx)$$

$$\psi = \forall x(Sx \to Px) \to \forall x(Sx \to Qx)$$

$$\models \phi \to \psi,$$

What is the interpolant?

$\models \phi \to \theta$ and $\models \theta \to \psi.$

Answer: $\theta = \forall x(Px \to Qx).$

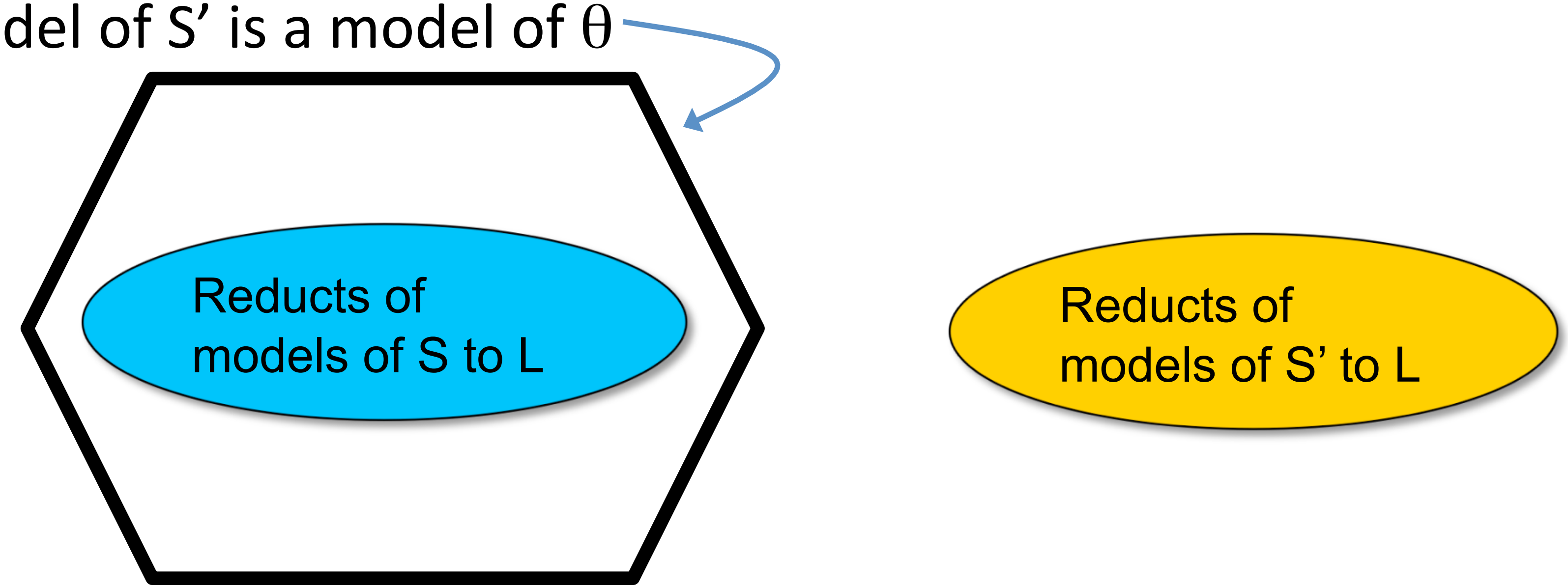
Separating *S* and *S'*

S a theory in the vocabulary L_1

S' a theory in the vocabulary L_2

θ a sentence in the vocabulary $L=L_1 \cap L_2$

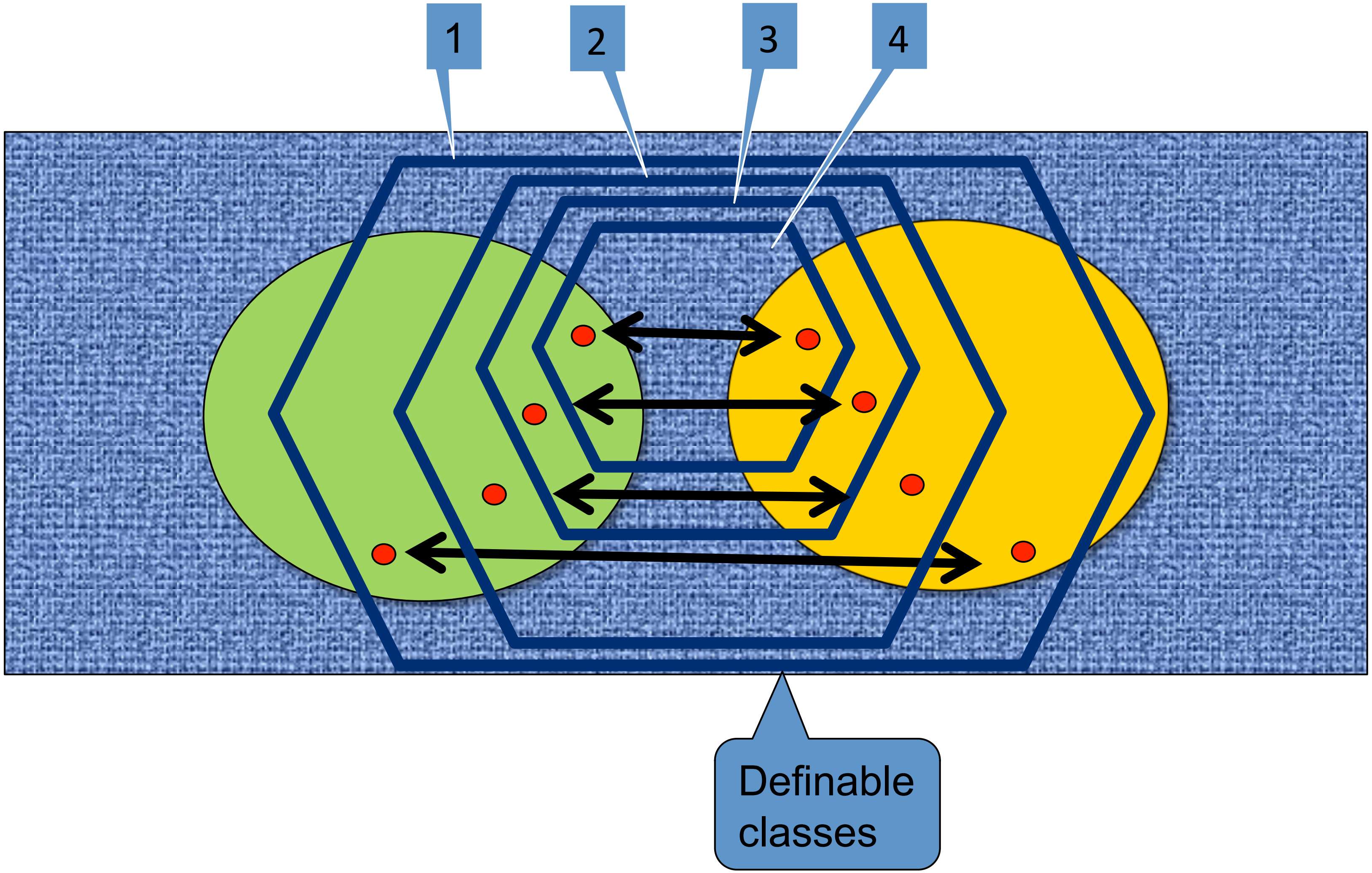
θ separates S and S' if every model of S is a model of θ but no model of S' is a model of θ



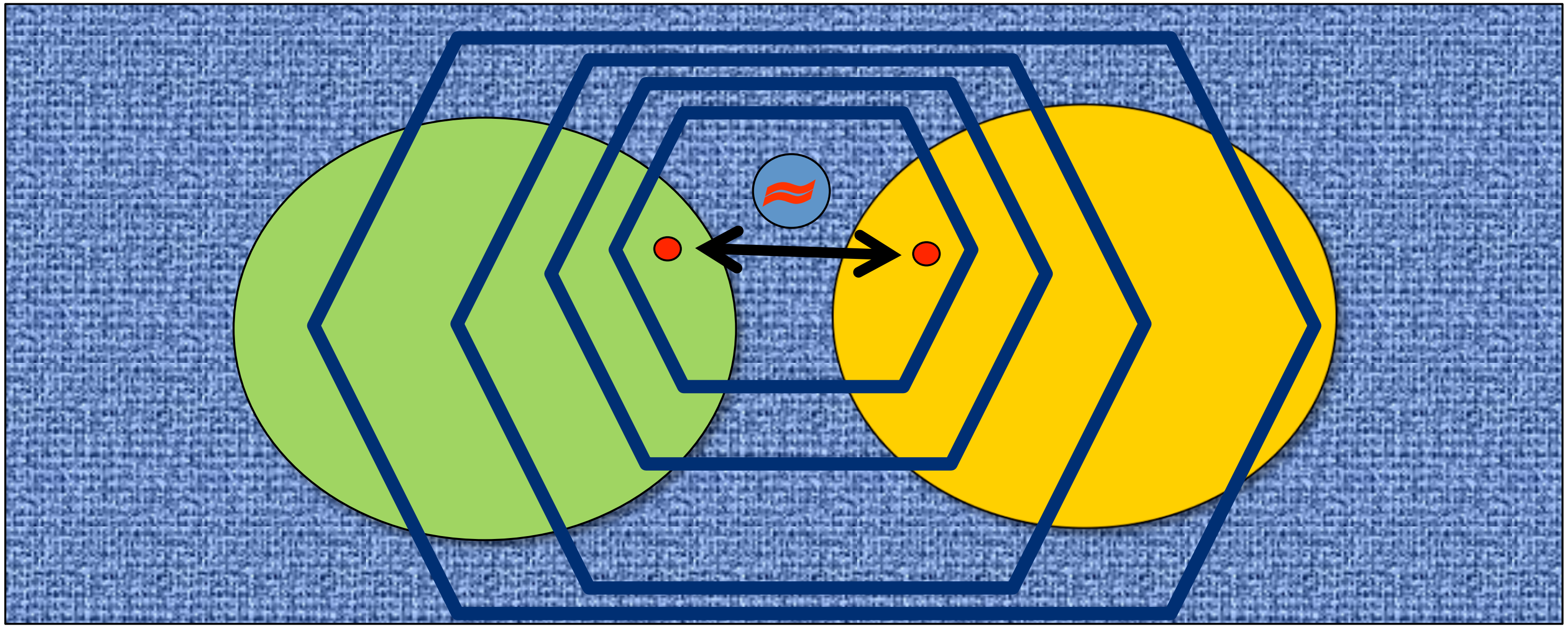
Interpolation strategy

- If φ and $\neg\psi$ cannot be separated, then II can play the whole MEG($\{\varphi, \neg\psi\}$,$L_1 \cup L_2$) game using this as a guiding principle:
  - She makes sure the sentences played by her divide into two parts (according to L_1 and L_2) that cannot be separated.
- Since this is a winning strategy, $\varphi \wedge \neg\psi$ has a model.
- Hence it cannot be that $\models \varphi \rightarrow \psi$!

Back-and-forth proof



Back-and-forth proof



Contradiction!

Lindström's Theorem

- First order logic is maximal wrt the Compactness Theorem and the Downward Löwenheim-Skolem Theorem.
- Versions exist for
  - Finite variable fragment (van Benthem, ten Cate, V.)
  - Modal logic (van Benthem)
  - Infinitary logic (Shelah)