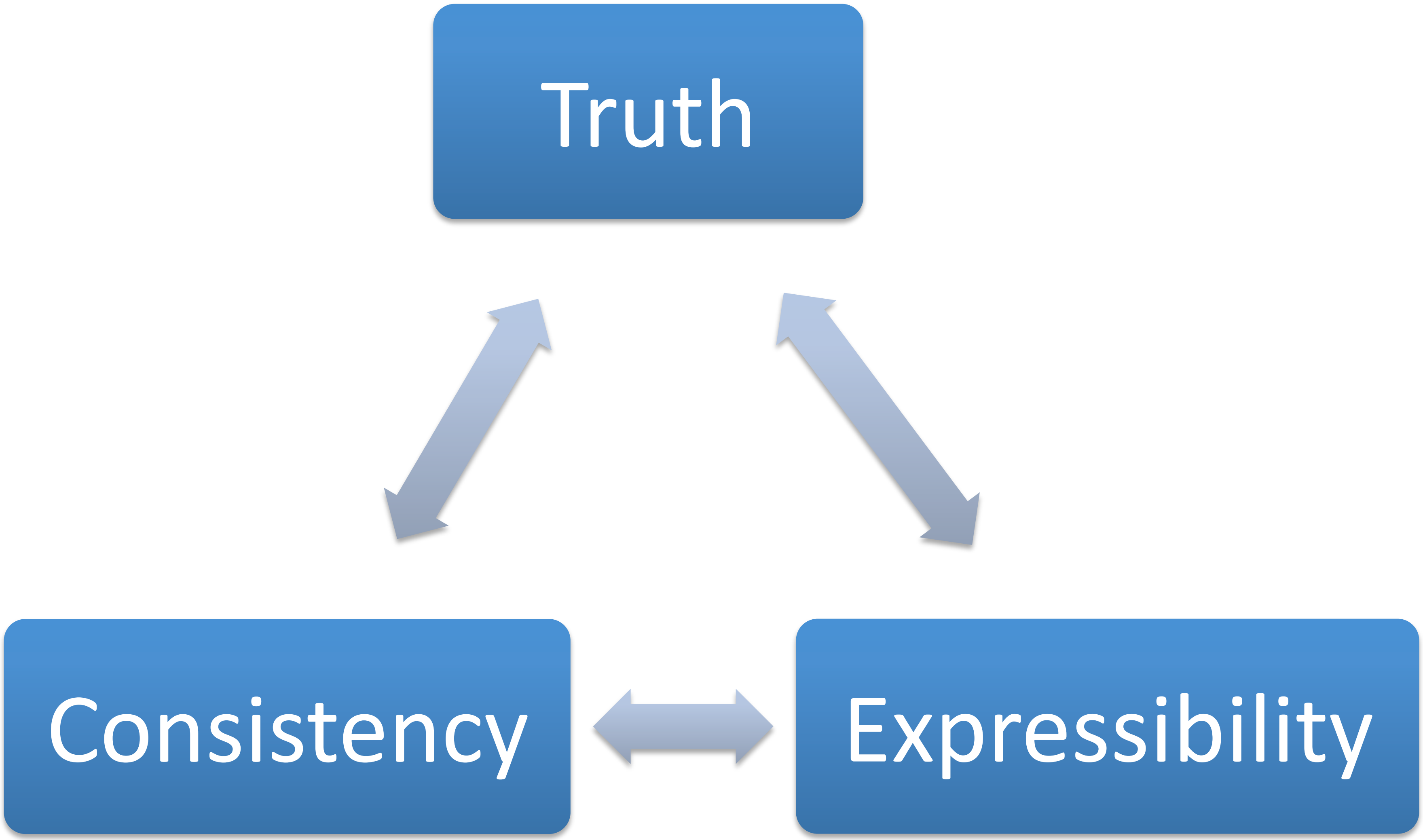
Lecture 1

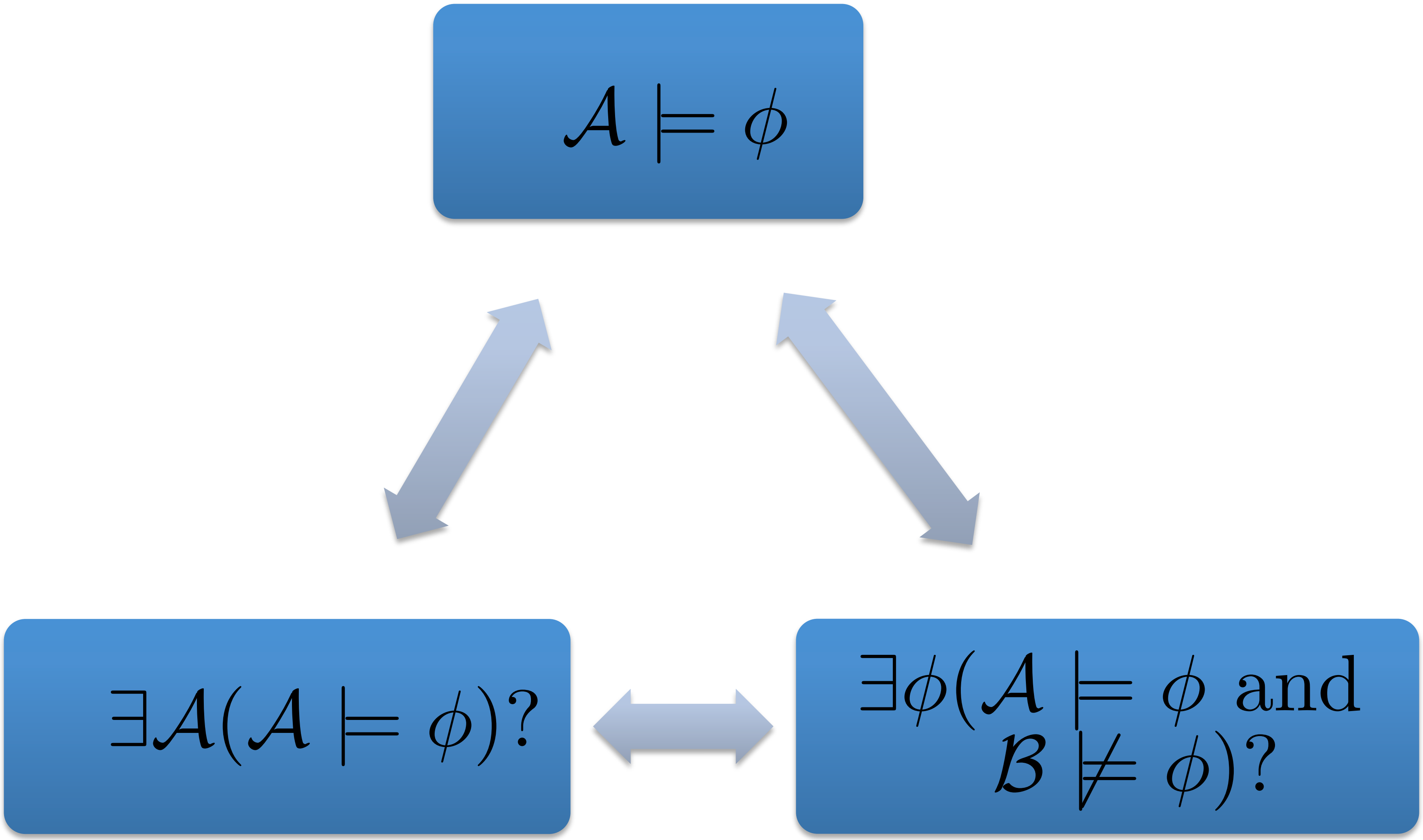
Jouko Väänänen

- What is the meaning of a given sentence?
- Can we say the same thing with a shorter sentence?

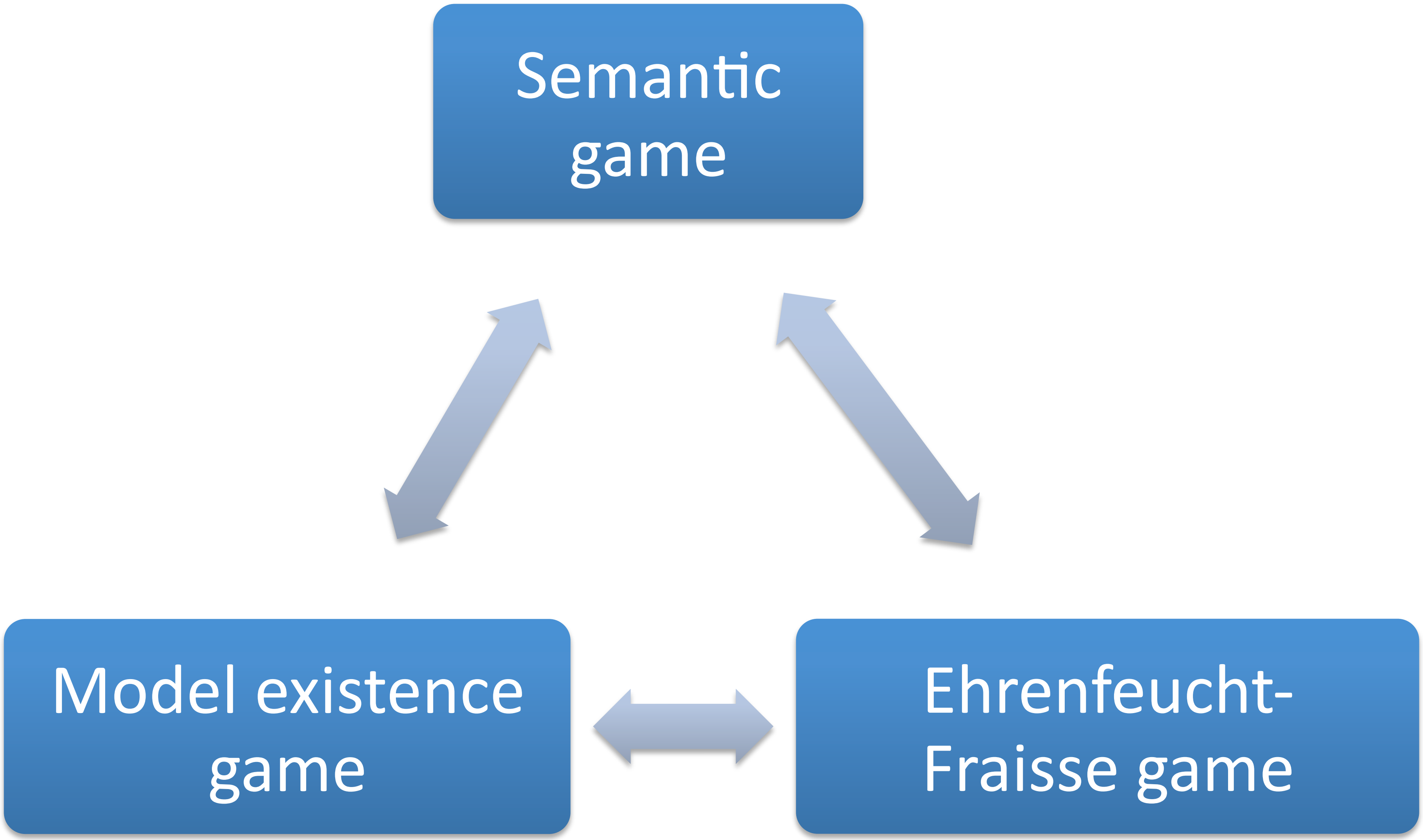
The three games of logic



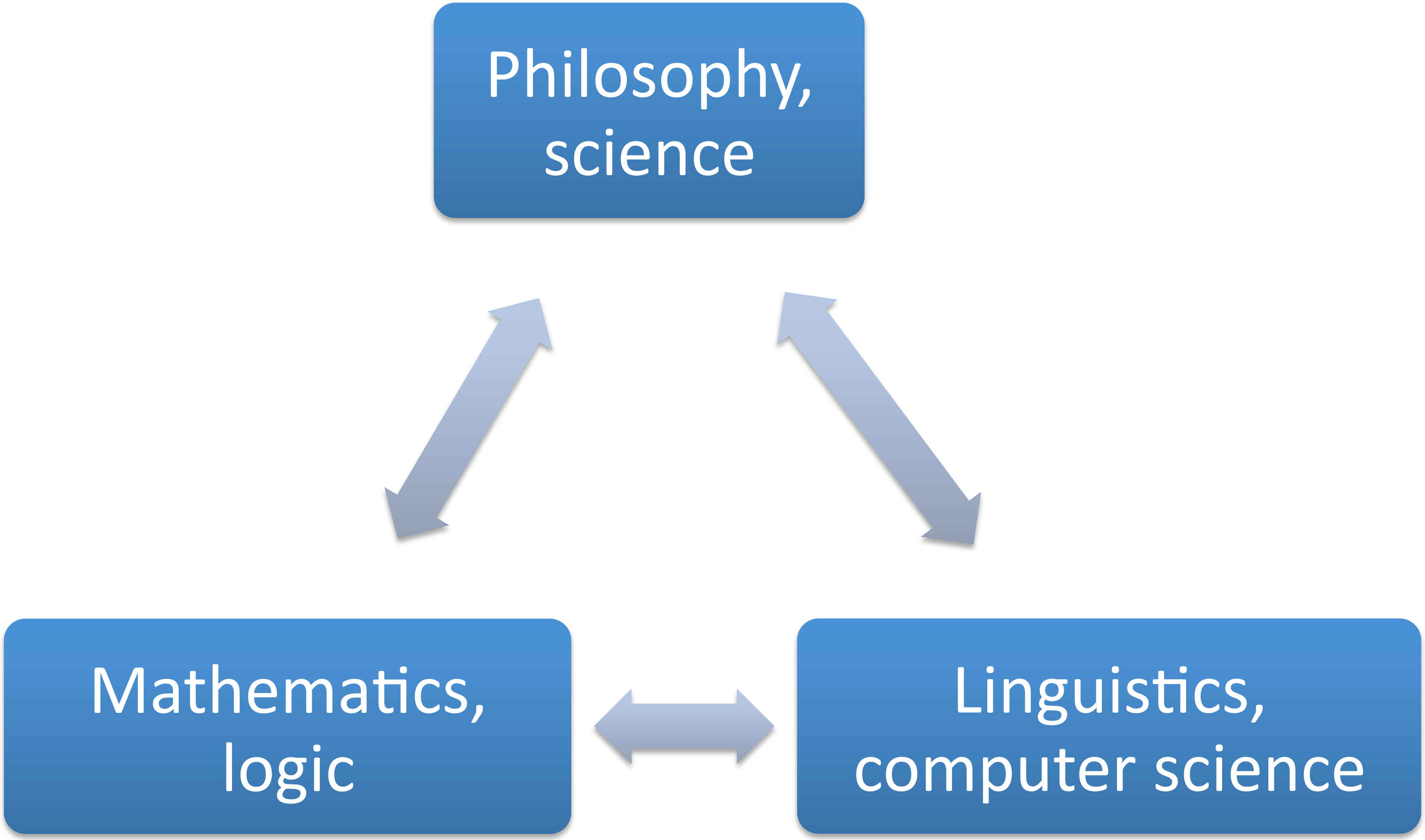
The three games of logic

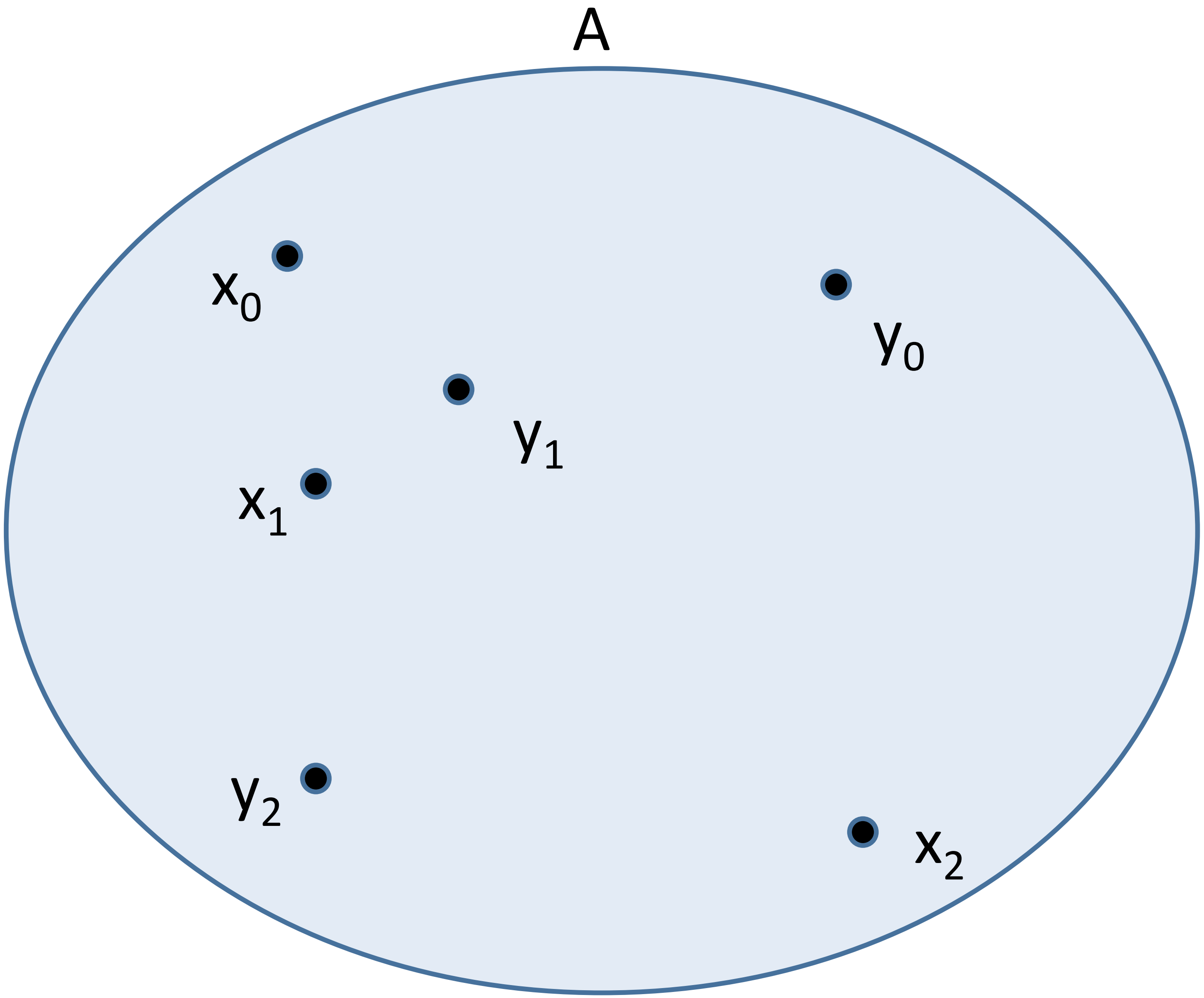


The three games of logic



The three games of logic





$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

Game

I	II
x_0	
	y_0
x_1	
	y_1
$\vdots$	
	$\vdots$
x_{n-1}	
	y_{n-1}

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

Winning

$$W \subseteq A^{2n}$$

$$\mathcal{G}_n(A, W)$$

$$(\mathbf{x}; \mathbf{y}) = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in W$$

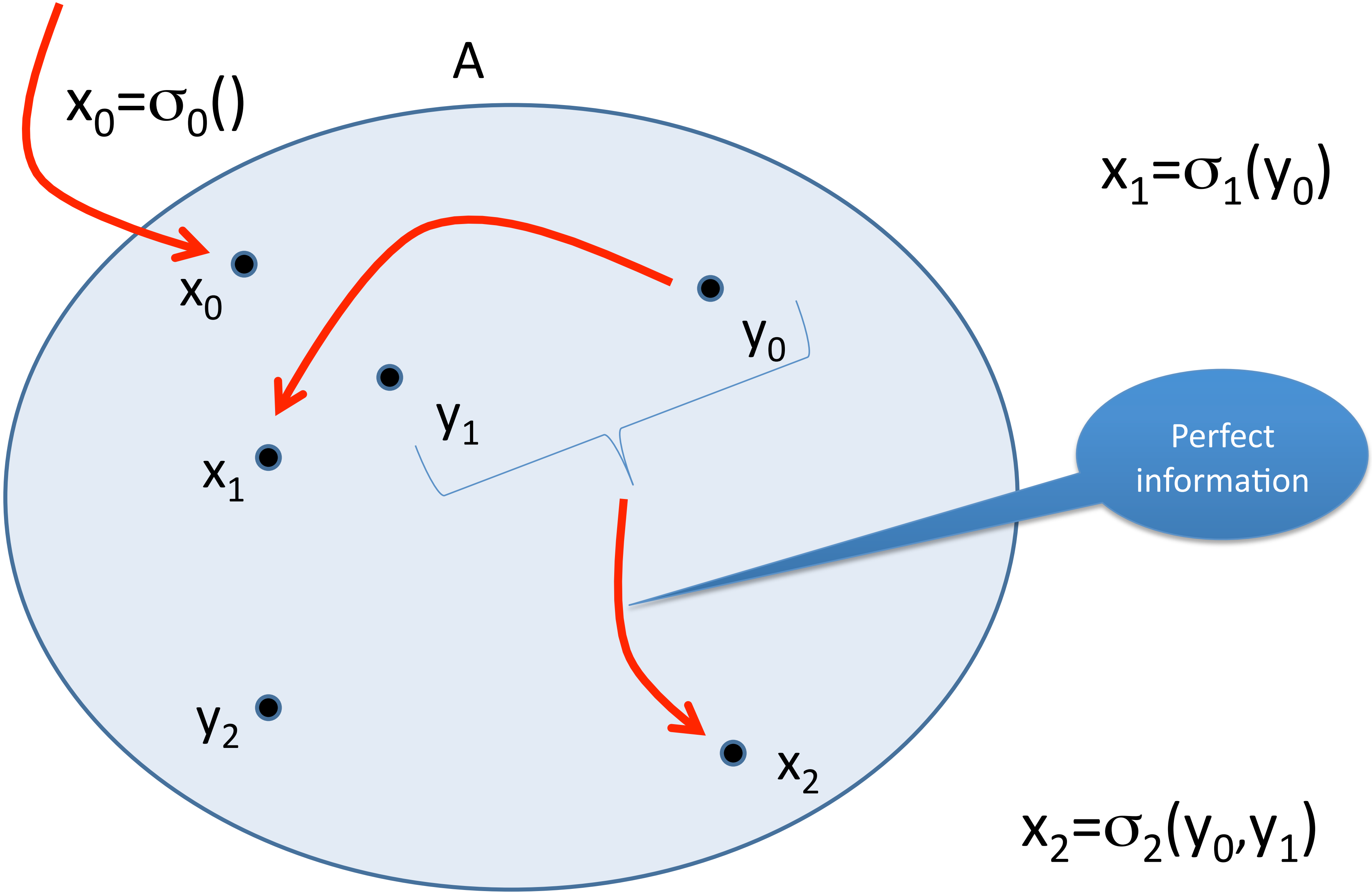
Strategy of player I

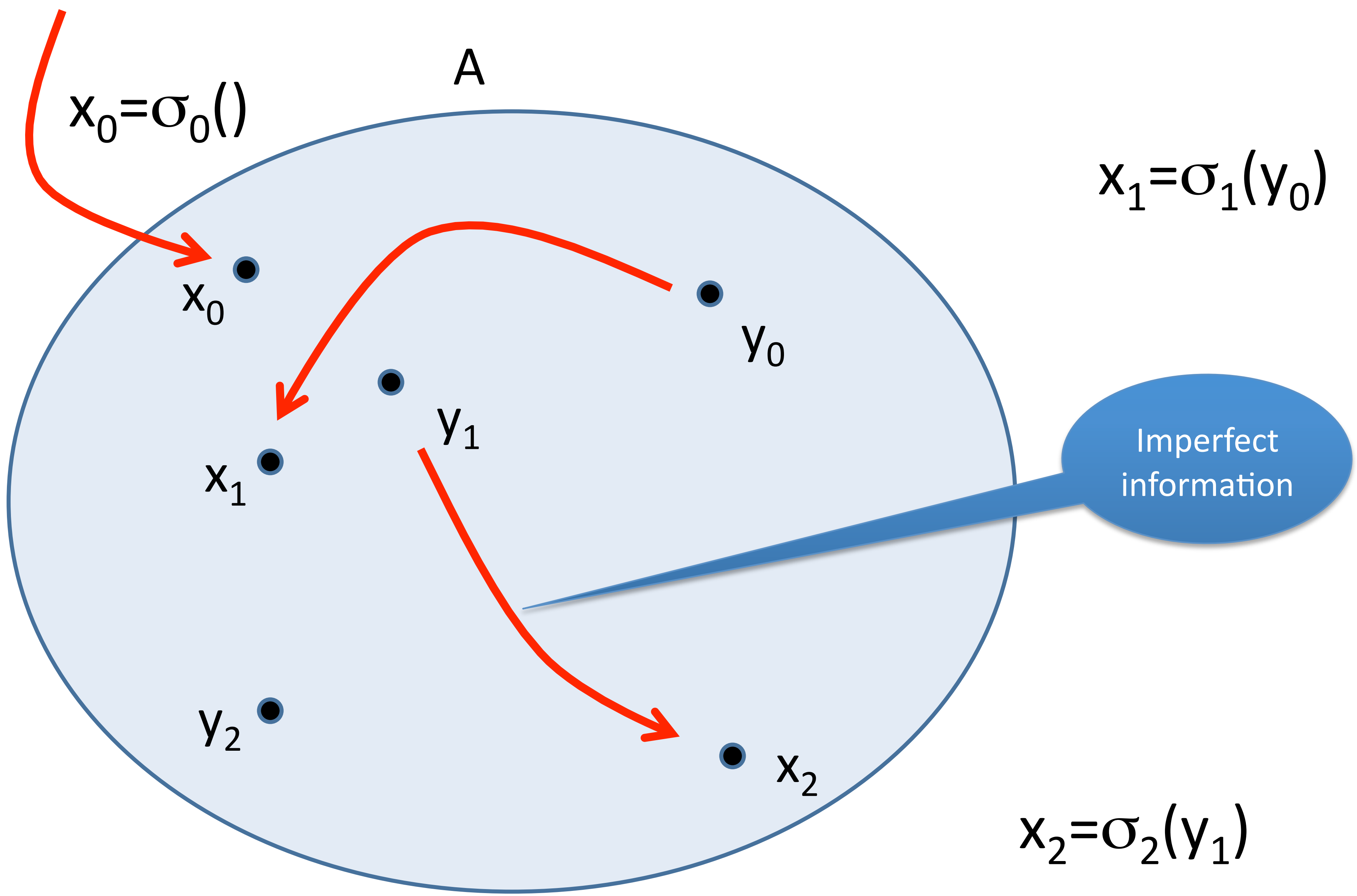
$$\sigma = (\sigma_0, \ldots, \sigma_{n-1})$$

$$\sigma_i : A^i \to A$$

Using a strategy:

$$x_i = \sigma_i(y_0, \ldots, y_{i-1})$$


A
$x_0=\sigma_0()$
$x_1=\sigma_1(y_0)$
x_0
y_0
y_1
x_1
Perfect information
y_2
x_2
$x_2=\sigma_2(y_0,y_1)$



A
x0=σ0()
x1=σ1(y0)
x0
y0
y1
x1
Imperfect information
y2
x2
x2=σ2(y1)

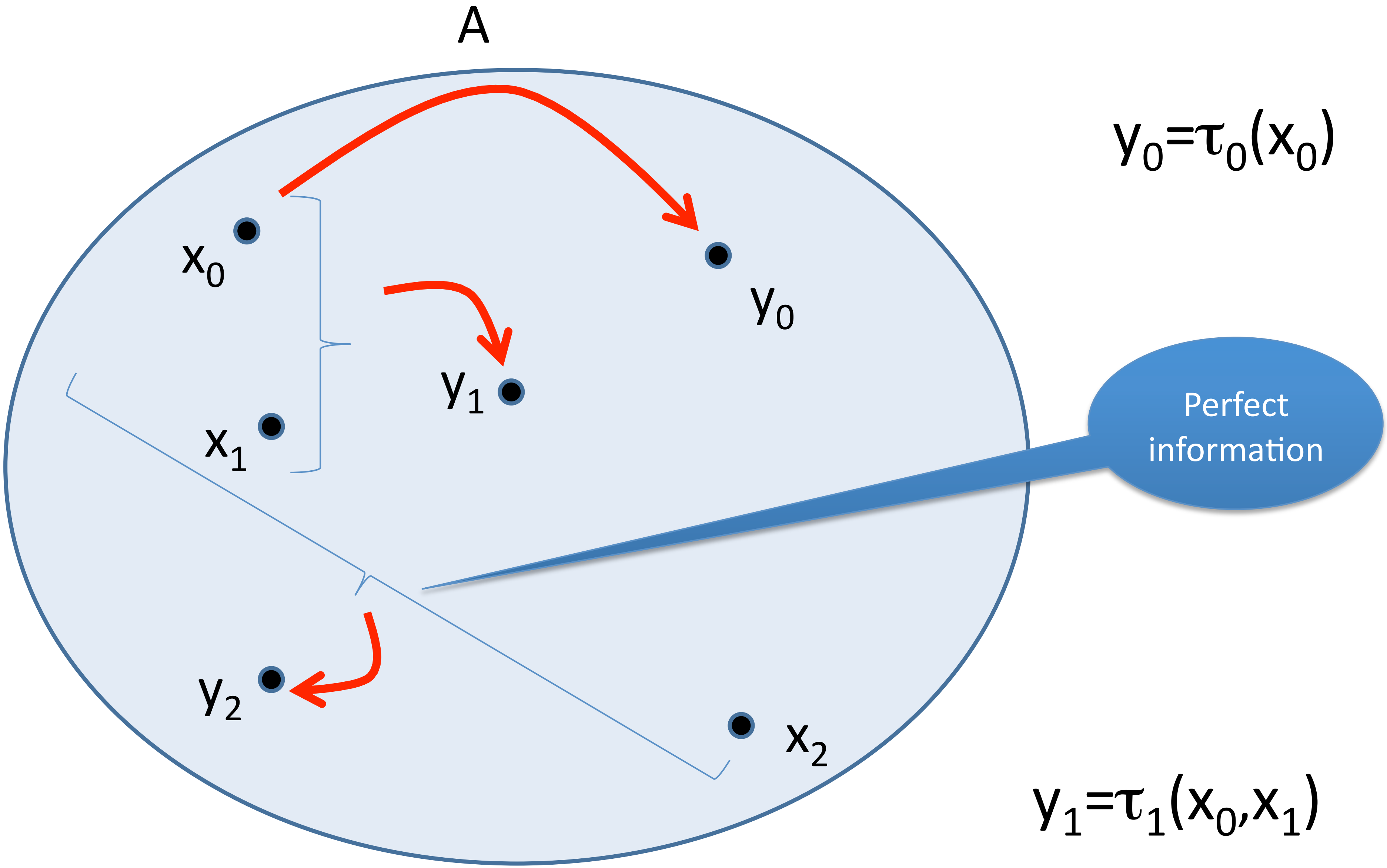
Strategy of player II

$$\tau = (\tau_0, \ldots, \tau_{n-1})$$

$$\tau_i : A^{i+1} \to A$$

Using a strategy:

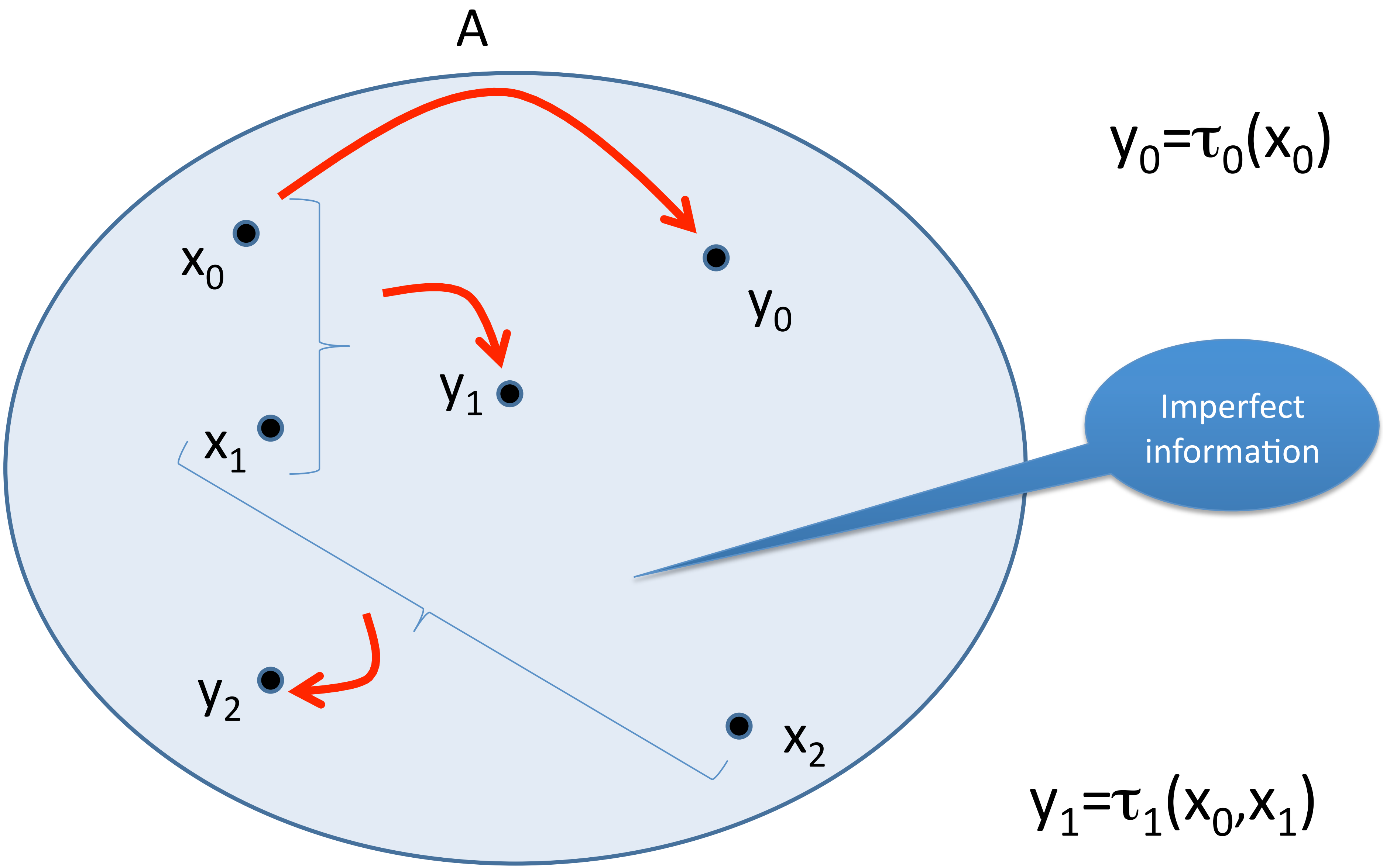
$$y_i = \tau_i(x_0, \ldots, x_i)$$



$y_0=\tau_0(x_0)$

$y_1=\tau_1(x_0,x_1)$

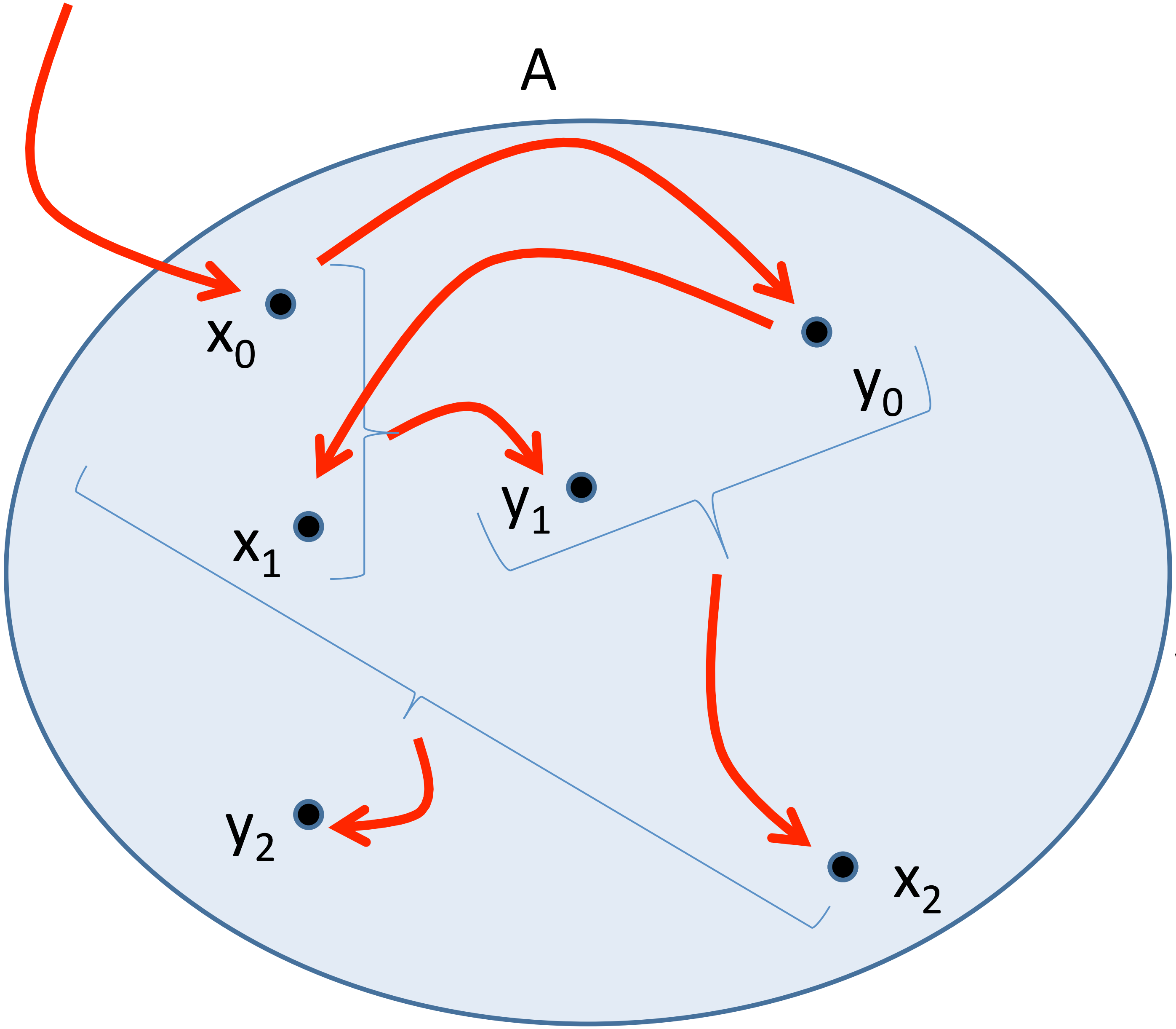
$y_2=\tau_1(x_0,x_1,x_2)$



$$y_0=\tau_0(x_0)$$

$$y_1=\tau_1(x_0,x_1)$$

$$y_2=\tau_1(x_1,x_2)$$



$$x_0=\sigma_0()$$

$$y_0=\tau_0(x_0)$$

$$x_1=\sigma_1(y_0)$$

$$y_1=\tau_1(x_0,x_1)$$

$$x_2=\sigma_2(y_0,y_1)$$

$$y_2=\tau_1(x_0,x_1,x_2)$$

Now we let two strategies play against each other:

Finite perfect information games are determined

Theorem 2.4.1 (Zermelo). *If A is any set, n is a natural number and $W \subseteq A^{2n}$, then the game $\mathcal{G}_n(A, W)$ is* determined, *i.e. one of the players has a winning strategy.*

Proof:

Case 1: Player I has a winning strategy. OK

Case 2: Player I does not have a winning strategy. Player II moves so that also after her move player I still does not have a winning strategy.

Proof

- Otherwise, whatever y player II moves, player I has a winning strategy g(y) for the rest of the game.
- Now I has a winning strategy already in the beginning of the game: Look at the move y of II and then use g(y).
- Perfect information needed, because I has to know the move y of II.

Non-determined game

- $y_1 = x_1$ to be chosen knowing only x_0
- II cannot have a winning strategy. How can she hit x_1 knowing only x_0.
- I cannot have a winning strategy: II may be lucky and I cannot prevent that.

Infinite game

I	II
x_0	
	y_0
x_1	
	y_1
$\vdots$	$\vdots$

Fig. 2.3. An infinite game

Infinite game

$$A^{\mathbb{N}}$$

$$(x_0, x_1, \ldots)$$

$$\mathcal{G}_\omega(A, W)$$

$$(\mathbf{x}; \mathbf{y}) = (x_0, y_0, x_1, y_1, \ldots)$$

$$(x_0, y_0, x_1, y_1, \ldots) \in W$$

Strategy in infinite game

$$\sigma = (\sigma_0, \sigma_1, \ldots)$$

$$x_i = \sigma_i(y_0, \ldots, y_{i-1})$$

$$\tau = (\tau_0, \tau_1, \ldots)$$

$$y_i = \tau_i(x_0, \ldots, x_i)$$

Closed game

$$(x_0, y_0, x_1, y_1, \ldots) \in W$$

if every initial segment of the play has **some** continuation in W.

In a closed game II wins if at any moment
she has at least one winning continuation.

Open game

$$(x_0, y_0, x_1, y_1, \ldots) \in W$$

implies the existence of $n \in \mathbb{N}$ such that

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x'_n, y'_n, x'_{n+1}, y'_{n+1}, \ldots) \in W$$

for all $x'_n, y'_n, x'_{n+1}, y'_{n+1}, \ldots \in A.$

Examples of infinite games

- $x_0 = 5$.
- $y_0 = 9$.
- Some $x_n = 0$.
- Some $y_n = 100$.
- For all n: $x_n = y_n$
- From some n onwards $y_m = 0$
- $y_n = 0$ for infinitely many n
- $y_n = 0$ for all n that are powers of 2

Gale-Stewart Theorem

Theorem 2.5.1 **(Gale-Stewart [8]).** *If A is any set and $W \subseteq A^{\mathbb{N}}$ is open or closed, then the game $\mathcal{G}_\omega(A, W)$ is determined.*

Proof:

Case 1: Player I has a winning strategy. OK

Case 2: Player I does not have a winning strategy. Player II plays so that also after her move player I still does not have a winning strategy. Since W is closed, player II wins.

Now from games to logic

Vocabulary

A vocabulary is a set L of

predicate symbols P,Q,R,...

function symbols f,g,h,...

constant symbols c,d,e,...

Arity function:

$$\#_L : L \to \mathbb{N}$$

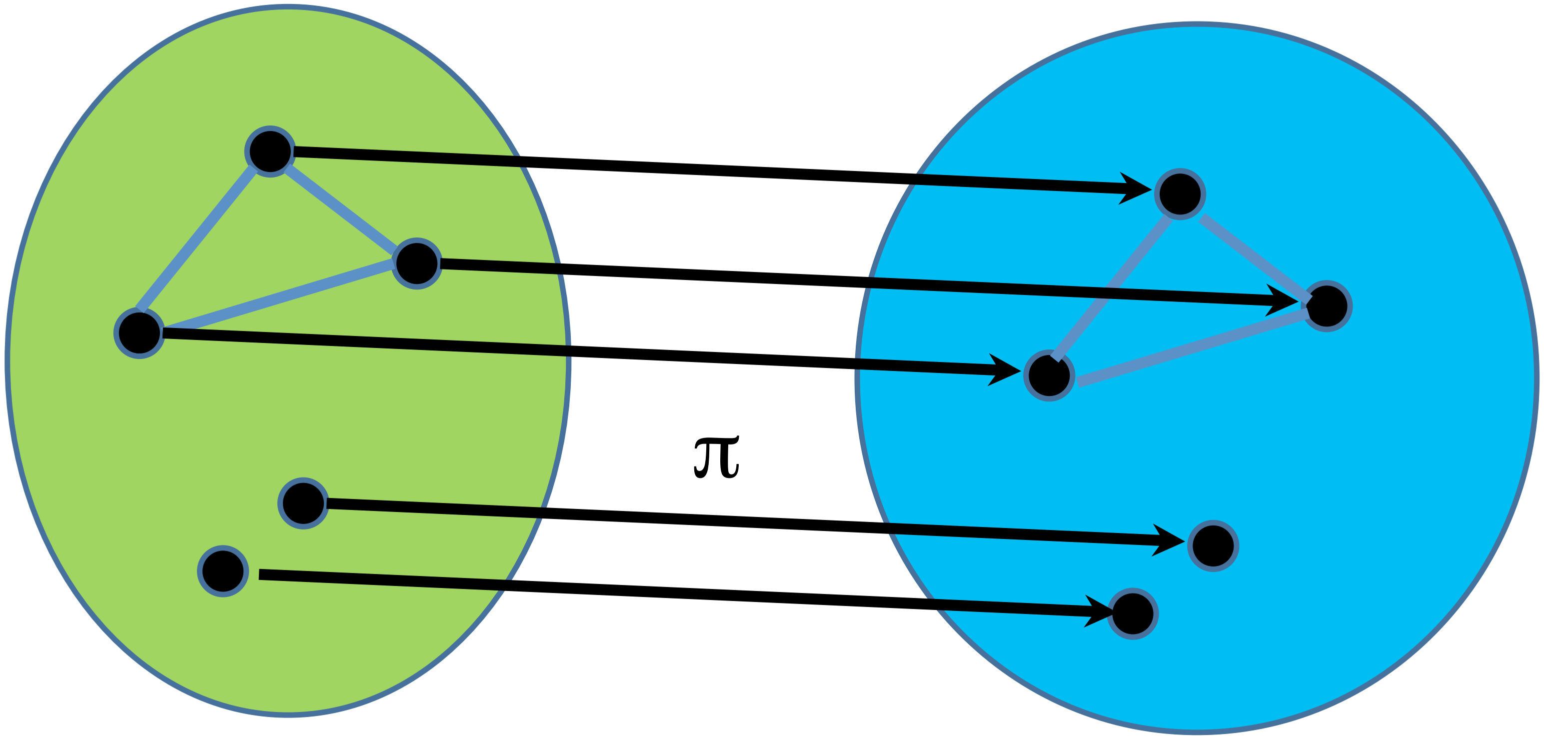
Model

- A model (or structure) M, for a vocabulary L is a non-empty set M, called the universe of M, and:
  - A subset P^M of M^n for every unary predicate symbol P in L of arity n
  - A function f^M of M^n into M for every function symbol f in L of arity n
  - An element c^M of M for every constant symbol c in L.

Examples

- Graphs
- Groups
- Unary structures
- Ordered sets
- Equivalence relations
- Fields

Isomorphism



Isomorphism defined

Definition 4.1.2. *L–structures $\mathcal{M}$ and $\mathcal{M}'$ are* isomorphic, *if there is a bijection*

$$\pi : M \to M'$$

such that

1. *For all $a_1, \ldots, a_{\#_L(R)} \in M$:*

$$(a_1, \ldots, a_{\#_L(R)}) \in R^{\mathcal{M}} \iff (\pi(a_1), \ldots, \pi(a_{\#_L(R)})) \in R^{\mathcal{M}'}$$

2. *For all $a_1, \ldots, a_{\#_L(f)} \in M$:*

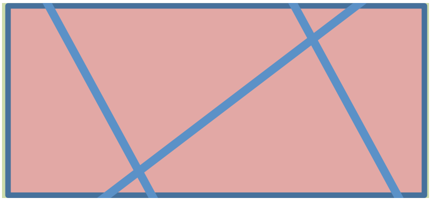
$$f^{\mathcal{M}'}(\pi(a_1), \ldots, \pi(a_{\#_L(f)})) = \pi(f^{\mathcal{M}}(a_1, \ldots, a_{\#_L(f)})).$$

3. $\pi(c^{\mathcal{M}}) = c^{\mathcal{M}'}$.

In this case we say that π is an isomorphism $\mathcal{M} \to \mathcal{M}'$

$$\pi : \mathcal{M} \cong \mathcal{M}'.$$

Substructure



Definition 4.2.1. *An L-structure $\mathcal{M}$ is a* substructure *of another L-structure $\mathcal{M}'$, in symbols $\mathcal{M} \subseteq \mathcal{M}'$, if:*

1. $M \subseteq M'$
2. $R^{\mathcal{M}} = R^{\mathcal{M}'} \cap M^n$ *if $R \in L$ is an n-ary predicate symbol.*
3. $f^{\mathcal{M}} = f^{\mathcal{M}'} \restriction M^n$ *if $f \in L$ is an n-ary function symbol.*
4. $c^{\mathcal{M}} = c^{\mathcal{M}'}$ *if $c \in L$ is a constant symbol.*

Generated substructure

Lemma 4.2.1. *Suppose L is a vocabulary, $\mathcal{M}$ an L-structure and $X \subseteq M$. Suppose furthermore that either L contains constant symbols or $X \neq \emptyset$. There is a unique L-structure $\mathcal{N}$ such that:*

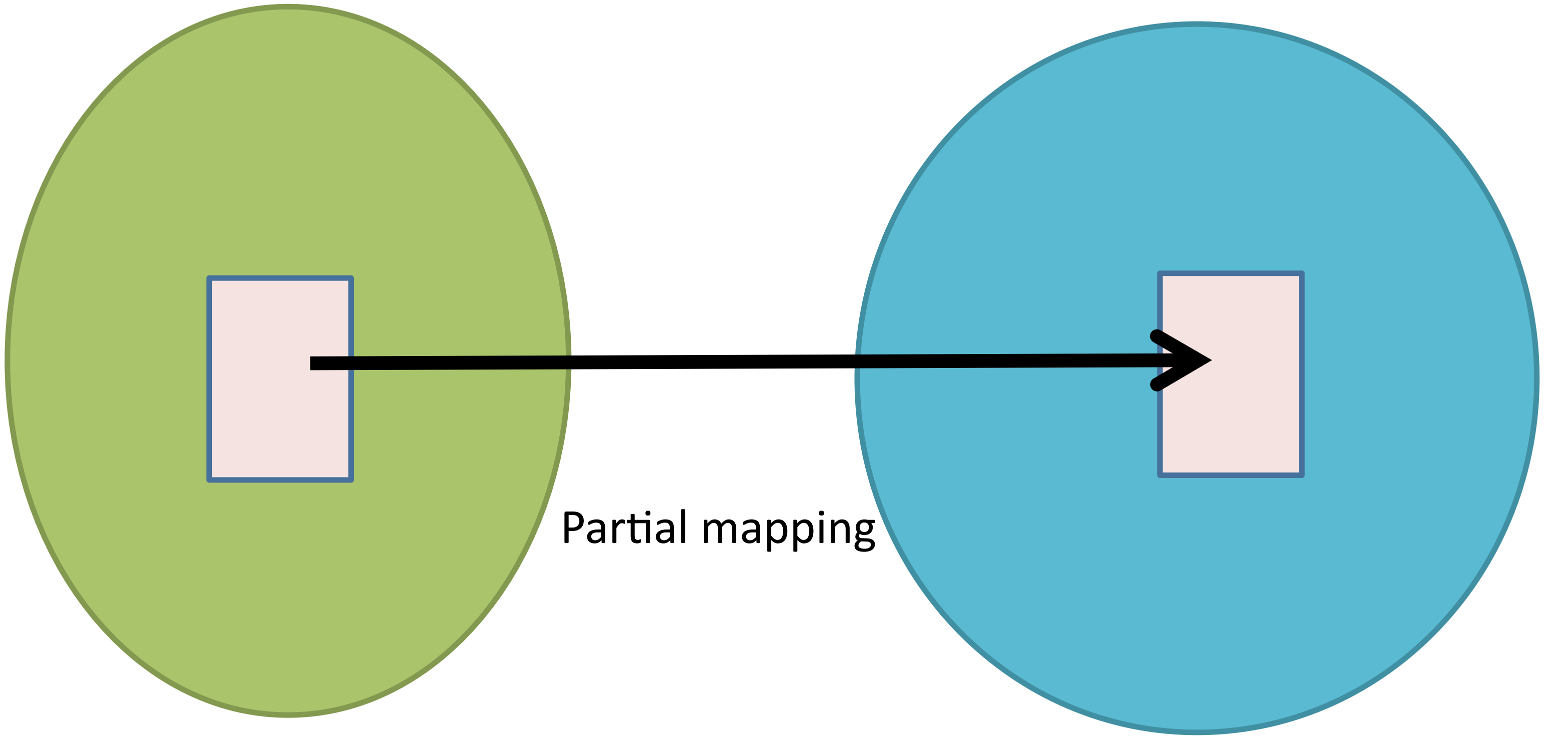
1. $\mathcal{N} \subseteq \mathcal{M}$.
2. $X \subseteq N$.
3. If $\mathcal{N}' \subseteq \mathcal{M}$ and $X \subseteq N'$, then $\mathcal{N} \subseteq \mathcal{N}'$.

Proof. Let $X_0 = X \cup \{c^{\mathcal{M}} : c \in L\}$ and inductively

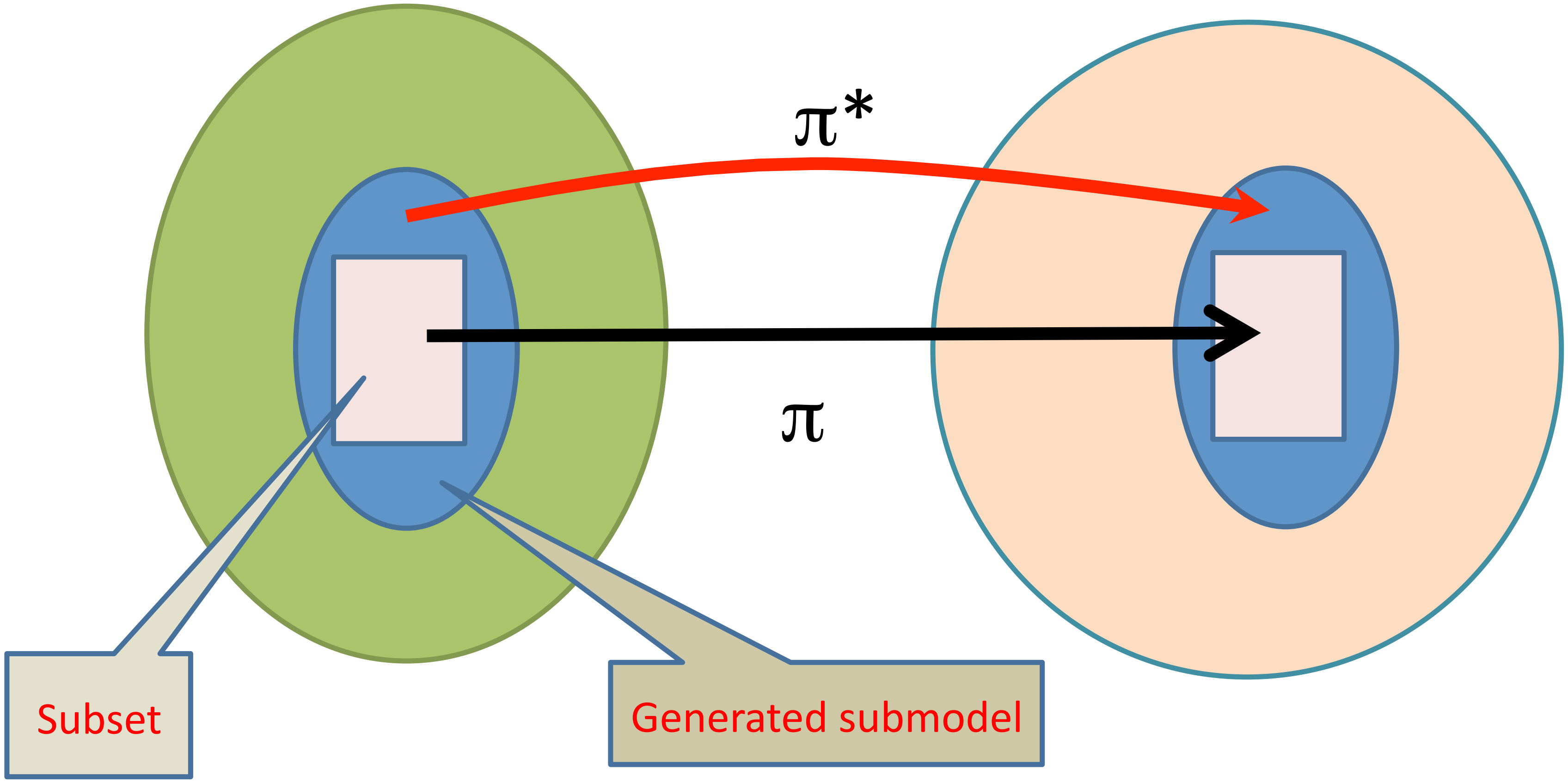
$$X_{n+1} = \{f^{\mathcal{M}}(a_1, \ldots, a_{\#_L(f)}) : a_1, \ldots, a_{\#_L(f)} \in X_n, f \in L\}.$$

It is easy to see that the set $N = \bigcup_{n \in \mathbb{N}} X_n$ is the universe of the unique structure $\mathcal{N}$ claimed to exist in the lemma. $\square$

Partial mappings



Lifting



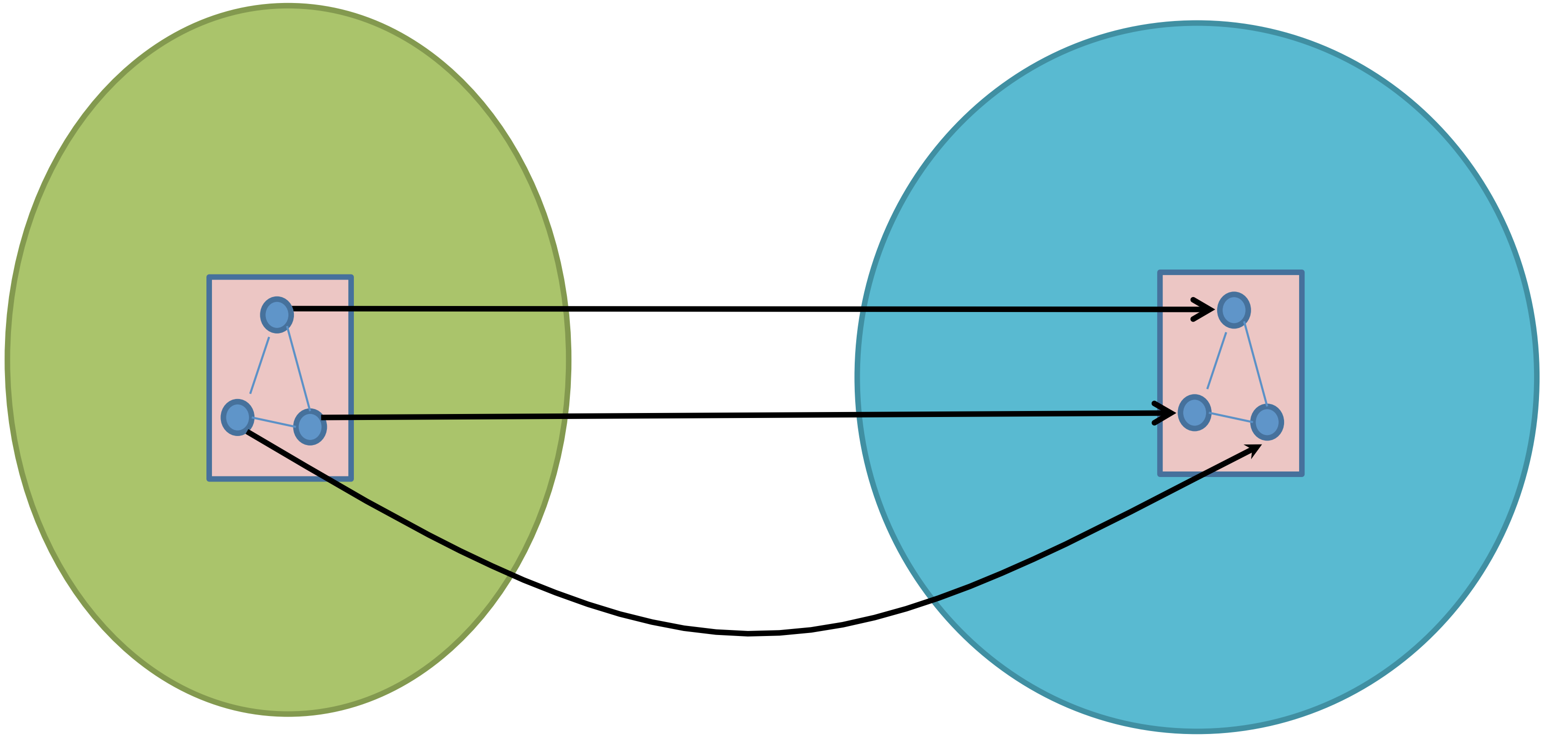
Lifting defined

Notation for the submodel generated by X

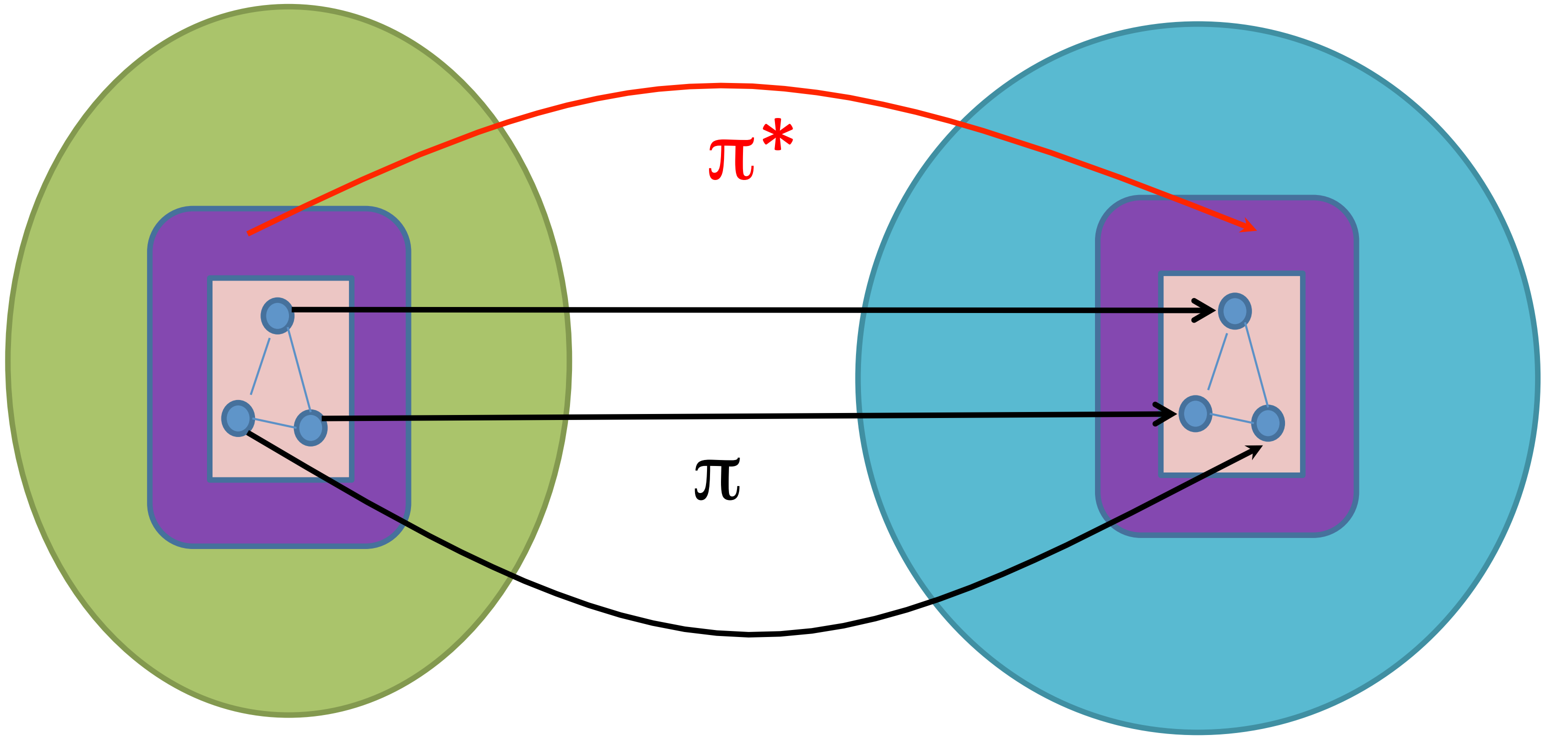
$$[X]_{\mathcal{M}}$$

Lemma 4.2.2. *Suppose L is a vocabulary. Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures and $\pi : M \to N$ is a partial mapping. There is at most one isomorphism $\pi^* : [\mathrm{dom}(\pi)]_{\mathcal{M}} \to [\mathrm{rng}(\pi)]_{\mathcal{N}}$ extending π.*

Partial isomorphism



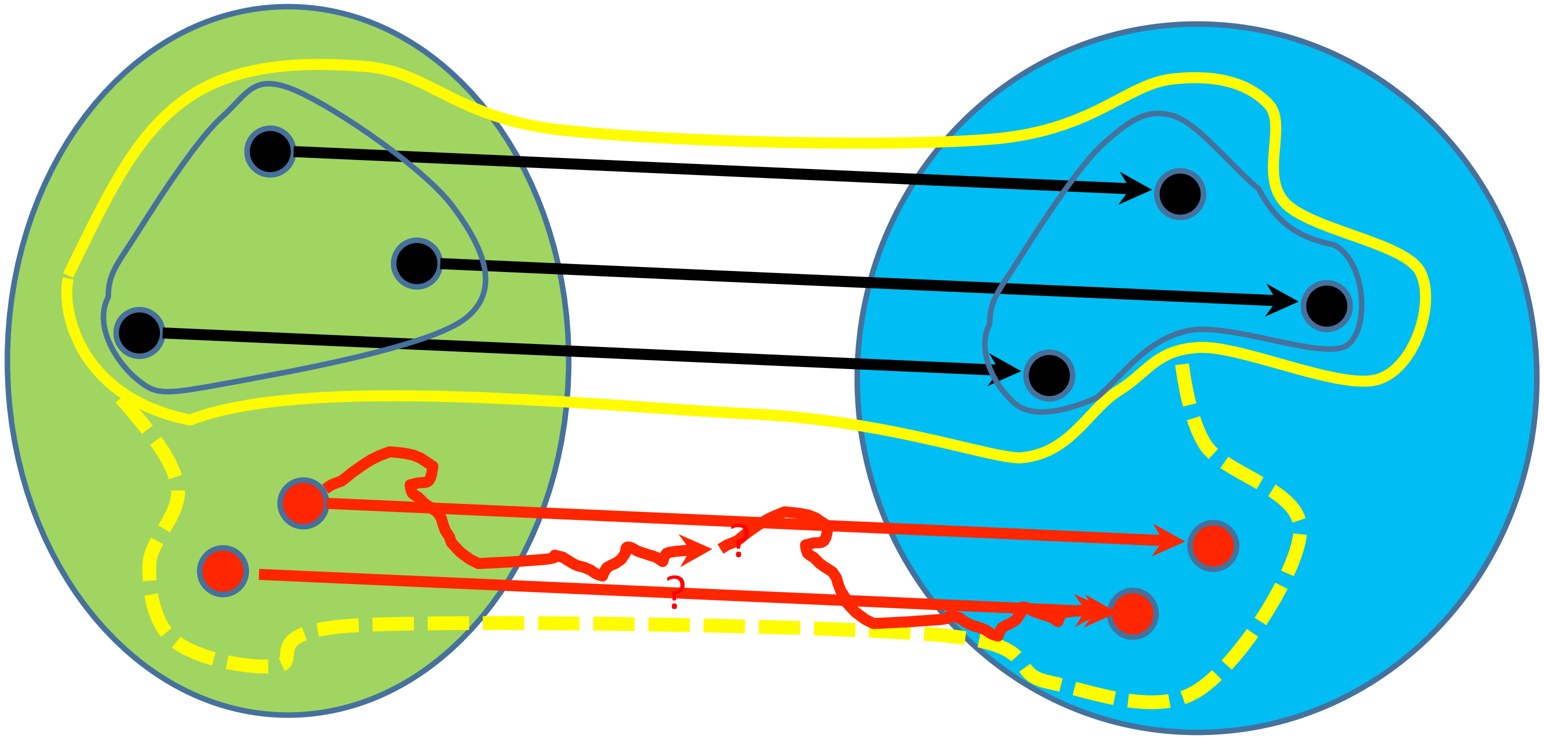
When there are functions



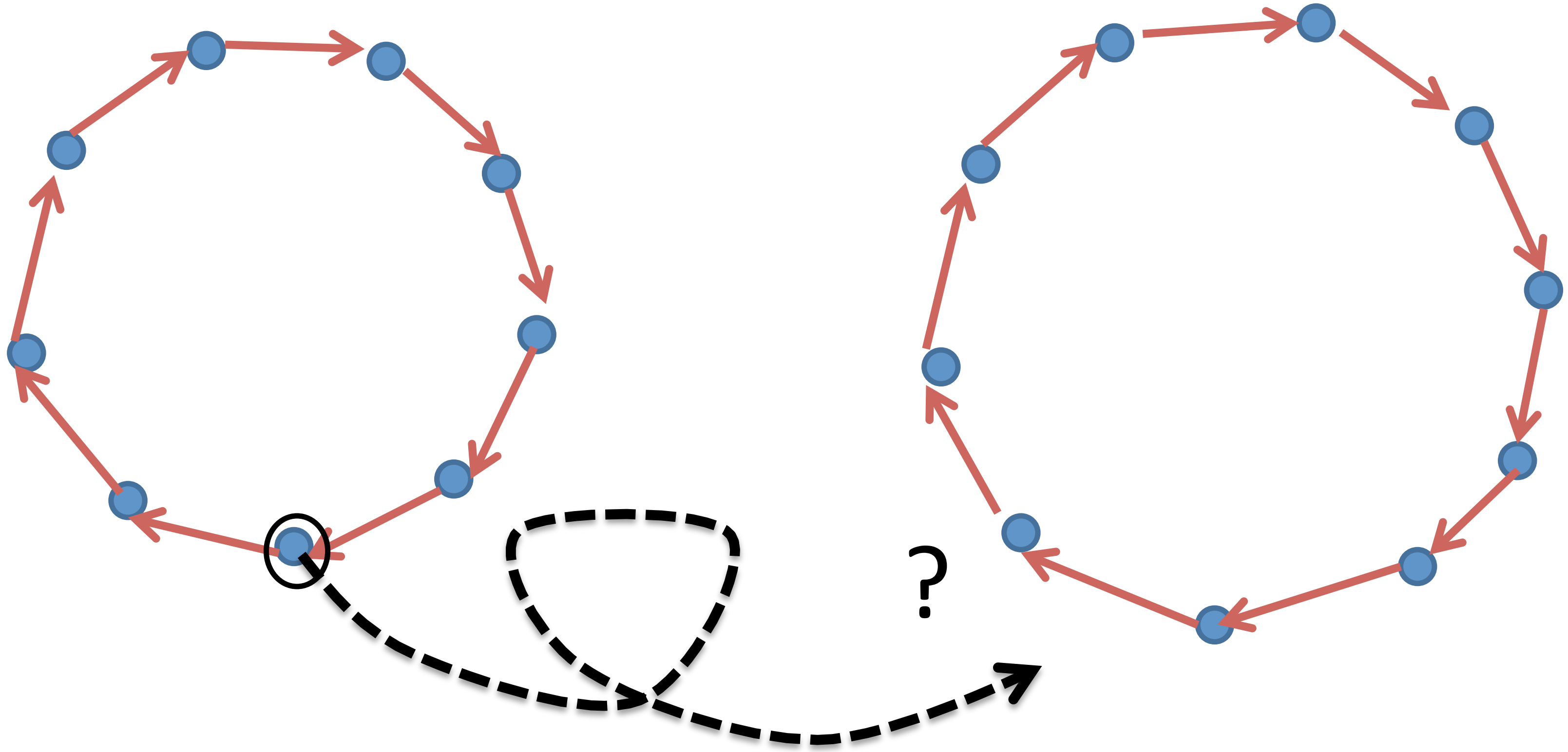
Partial isomorphism

Definition 4.3.1. *Suppose* L *is a vocabulary and* $\mathcal{M}, \mathcal{M}'$ *are* L*-structures. A partial mapping* $\pi : M \to M'$ *is a* partial isomorphism $\mathcal{M} \to \mathcal{M}'$ *if there is an isomorphism* $\pi^* : [\mathrm{dom}(\pi)]_{\mathcal{M}} \to [\mathrm{rng}(\pi)]_{\mathcal{M}'}$ *extending* π*. We use* $\mathrm{Part}(\mathcal{M}, \mathcal{M}')$ *to denote the set of partial isomorphisms* $\mathcal{M} \to \mathcal{M}'$*. If* $\mathcal{M} = \mathcal{M}'$ *we call* π *a* partial automorphism.

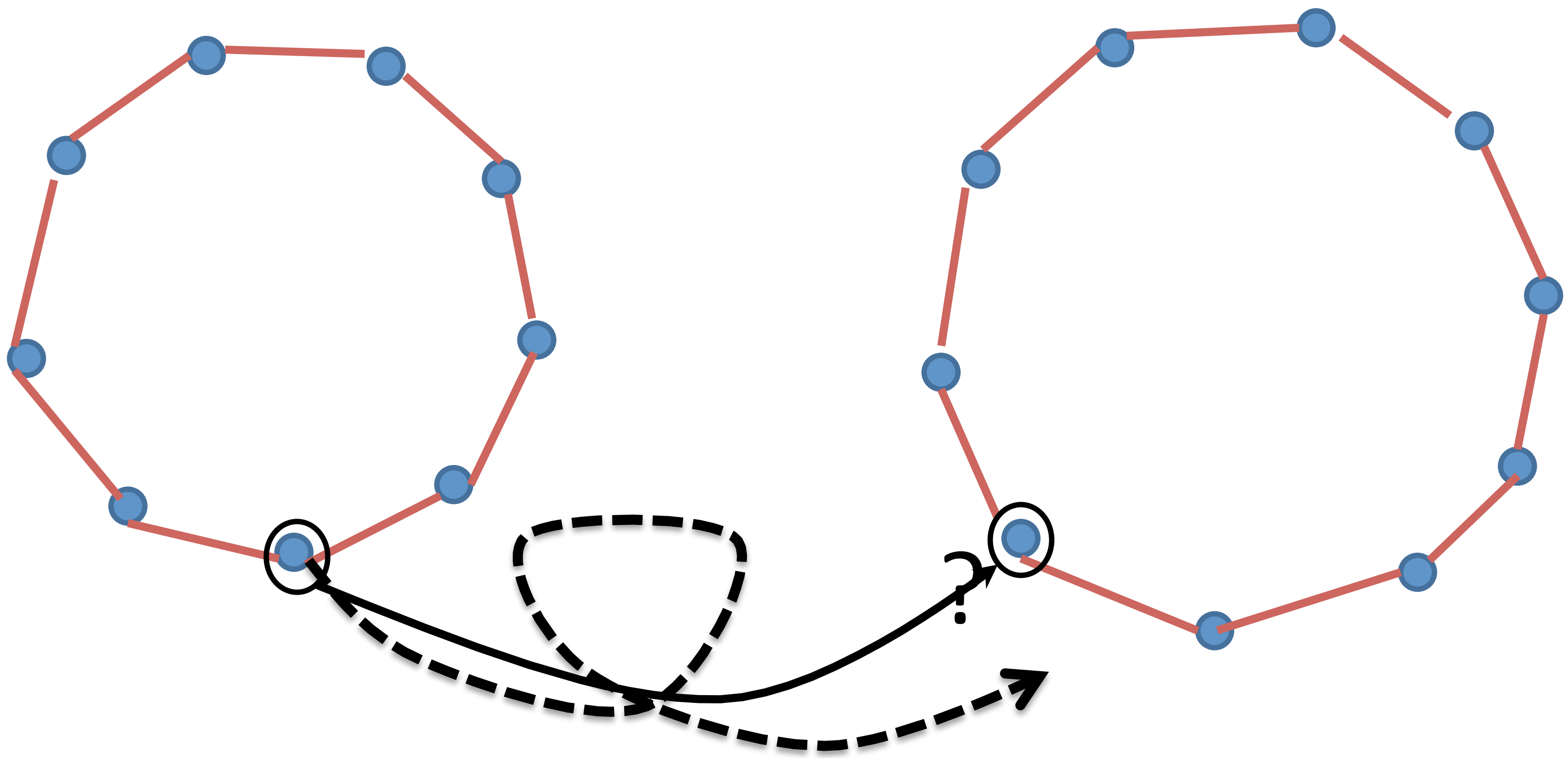
Extending a partial isomorphism



Two unary functions

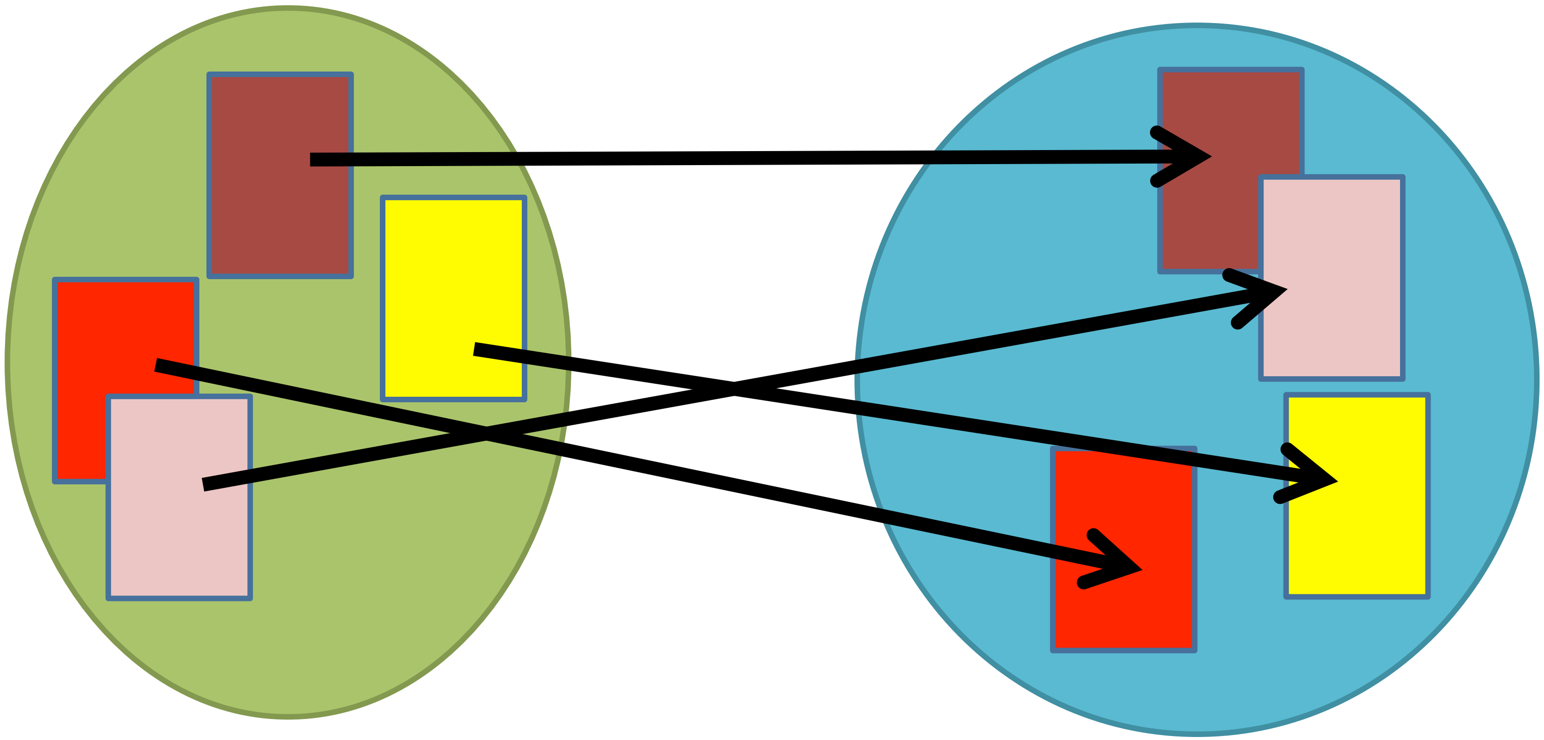


Two graphs

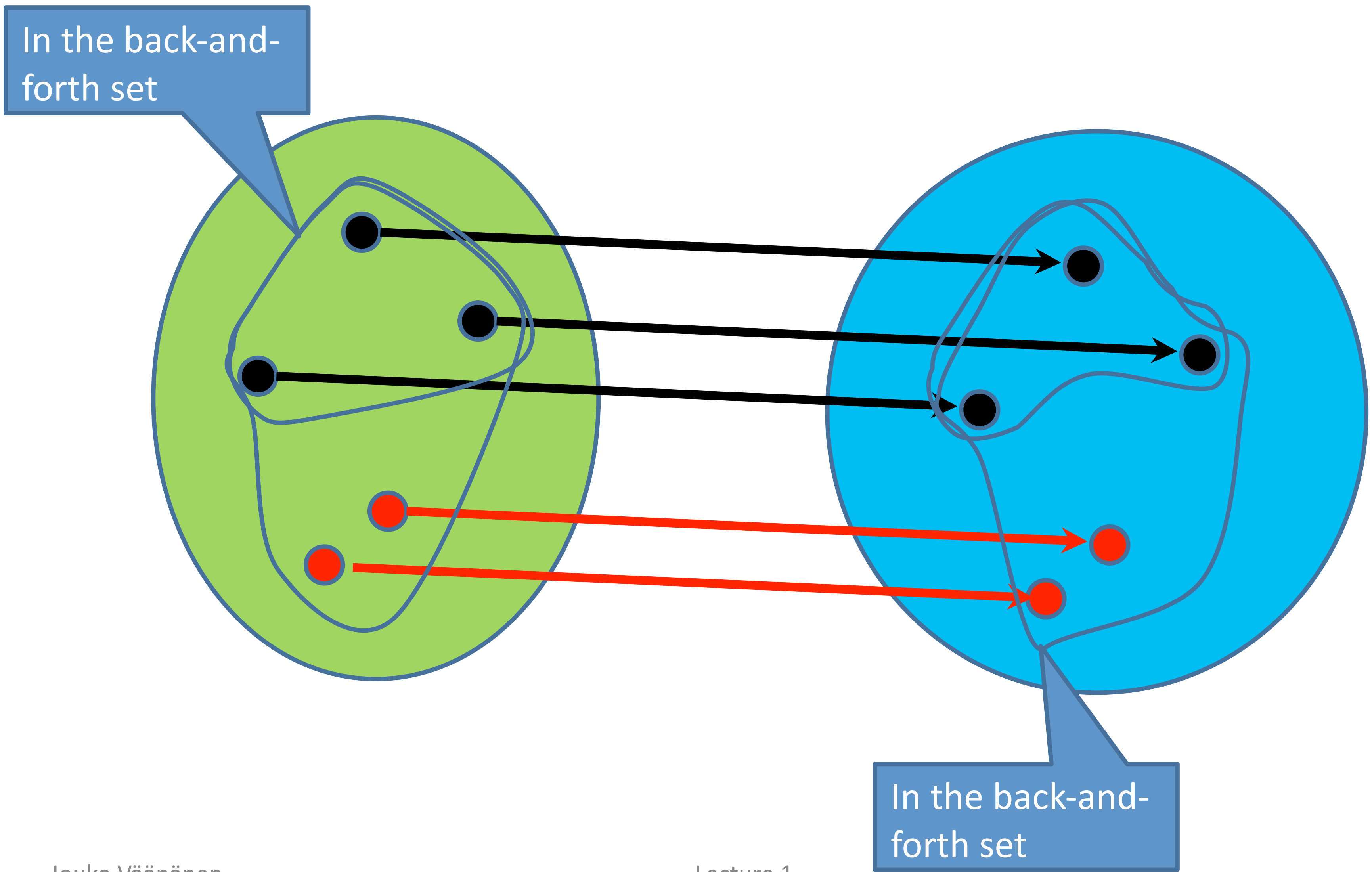


Back-and-forth set

A set of partial isomorphisms satisfying the sc. back-and-forth condition.



Back-and-forth condition



Back and forth set defined

Definition 2.3.2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are L-structures. A* back-and-forth set *for $\mathcal{A}$ and $\mathcal{B}$ is any non-empty set $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B})$ such that*

$$\forall f \in P \forall a \in A \exists g \in P (f \subseteq g \text{ and } a \in \mathrm{dom}(g)) \tag{2.8}$$

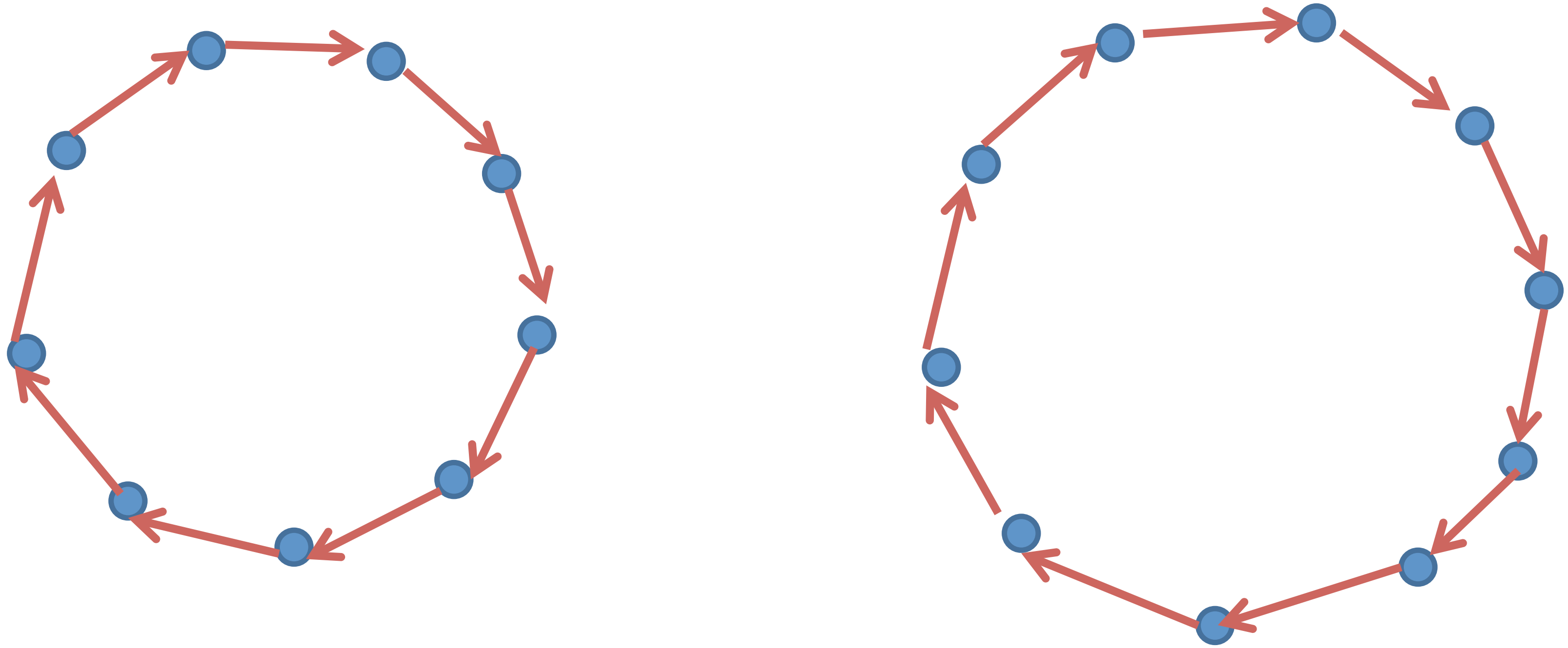
$$\forall f \in P \forall b \in B \exists g \in P (f \subseteq g \text{ and } b \in \mathrm{rng}(g)) \tag{2.9}$$

Partially isomorphic models

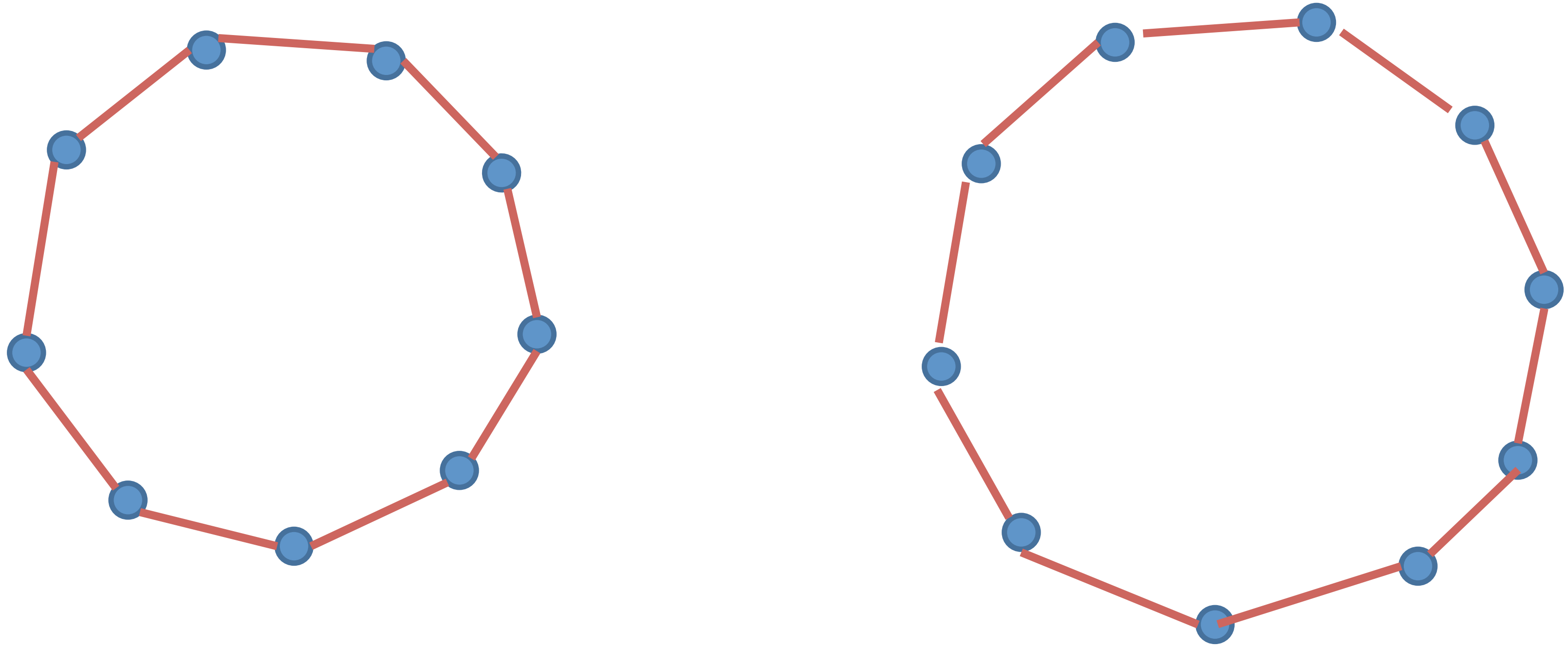
$$\mathcal{A} \simeq_p \mathcal{B}$$

Models are partially isomorphic if there is a back-and-forth set of partial isomorphisms between them.

Not partially isomorphic



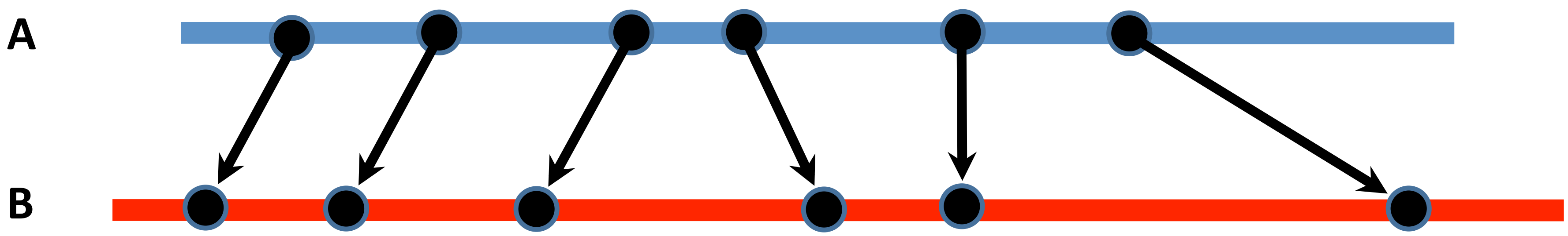
Not partially isomorphic



Dense total orders

- All dense total orders without endpoints are partially isomorphic.

$$P = \{f \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : \mathrm{dom}(f) \text{ is finite}\}$$

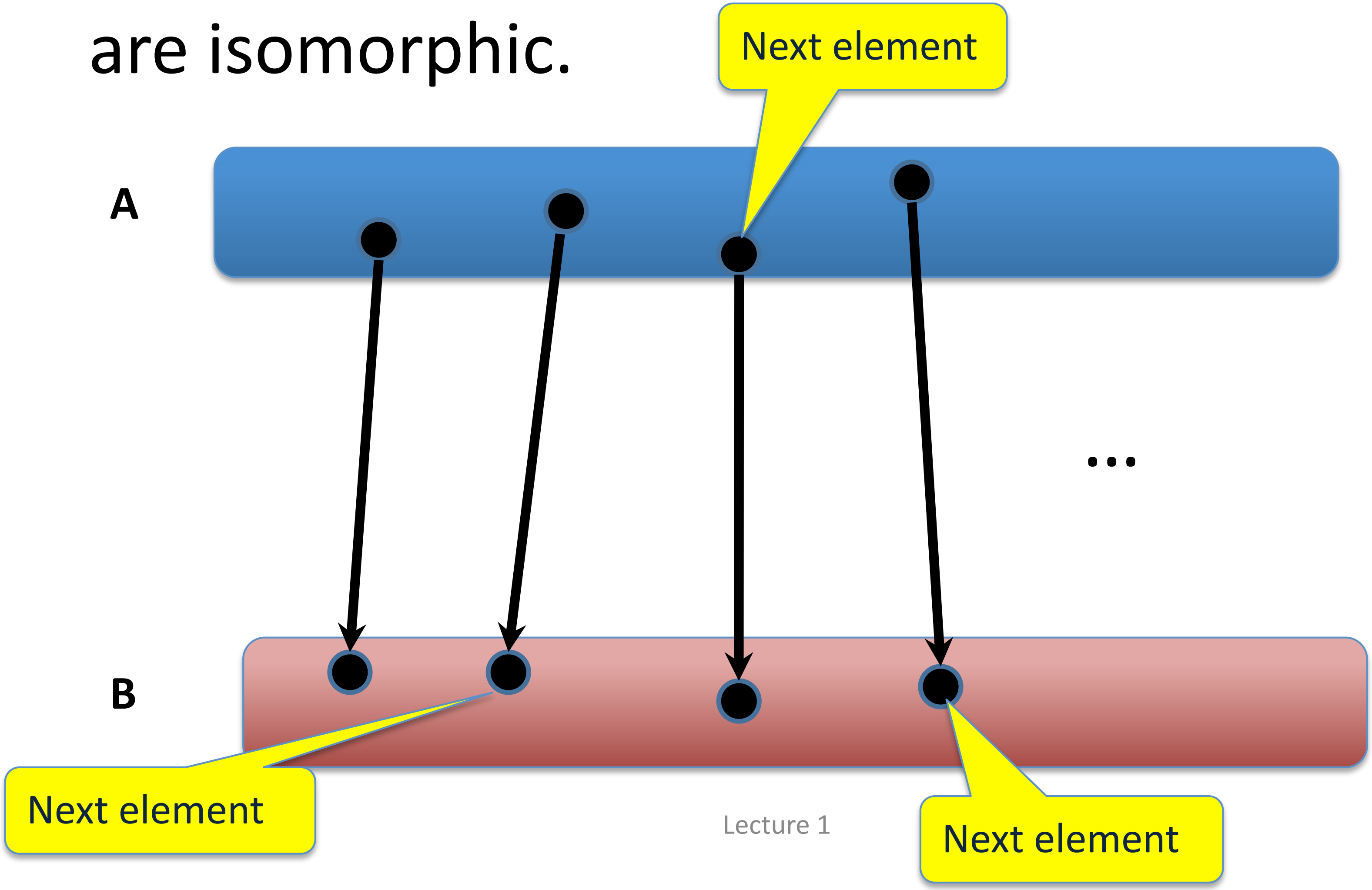


Proposition 4.3.2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are dense linear orders without endpoints. Then $\mathcal{A} \simeq_p \mathcal{B}$.*

Proof. Let $P = \{f \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : \mathrm{dom}(f) \text{ finite}\}$. It turns out that this straightforward choice works. Clearly, $P \neq \emptyset$. Suppose then $f \in P$ and $a \in A$. Let us enumerate f as $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ where $a_1 < \ldots < a_n$. Since f is a partial isomorphism, also $b_1 < \ldots < b_n$. Now we consider different cases. If $a < a_1$, we choose $b < b_1$ and then $f \cup \{(a, b)\} \in P$. If $a_i < a < a_{i+1}$, we choose $b \in B$ so that $b_i < b < b_{i+1}$ and then $f \cup \{(a, b)\} \in P$. If $a_n < a$, we choose $b > b_n$ and again $f \cup \{(a, b)\} \in P$. Finally, if $a = a_i$, we let $b = b_i$ and then $f \cup \{(a, b)\} = f \in P$. We have proved (4.8). Condition (4.9) is proved similarly. □

Countable models

- All countable partially isomorphic structures are isomorphic.



Proposition 2.3.1. *If $\mathcal{A} \simeq_p \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are countable, then $\mathcal{A} \cong \mathcal{B}$.*

Proof. Let us enumerate A as $(a_n : n < \omega)$ and B as $(b_n : n < \omega)$. Let P be a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. Since $P \neq \emptyset$, there is some $f_0 \in P$. We define a sequence $(f_n : n < \omega)$ of elements of P as follows: Suppose $f_n \in P$ is defined. If n is even, say $n = 2m$, let $y \in B$ and $f_{n+1} \in P$ such that $f_n \cup \{(a_m, y)\} \subseteq f_{n+1}$. If n is odd, say $n = 2m + 1$, let $x \in A$ and $f_{n+1} \in P$ such that $f_n \cup \{(x, b_m)\} \subseteq f_{n+1}$. Finally, let

$$f = \bigcup_{n=0}^{\infty} f_n.$$

Clearly, $f : \mathcal{A} \cong \mathcal{B}$. $\square$

Zero-one law

- Extension axioms
- Countable categoricity
- Glebskii et al, Fagin, zero-one law
  - Application of back-and-forth

Summary of Lecture 1

- Basic concepts about games, determinacy
- Partial isomorphism
- Back-and-forth set
- Next Lecture:
  - EF game
  - Characterization of (infinitary) elementary equivalence
  - Characterization of definability