EXAMPLE 11.3

Which of the following constant density flows are physically possible?

A. $u = \left(\frac{h^2(p_1 - p_2)}{2\mu L}\right)\left[1 - \left(\frac{y}{h}\right)^2\right]$, $v = 0$, and $w = 0$

B. $v_r = 0$, $v_\theta = 0$, and $v_z(r) = \left(\frac{R_P^2(p_1 - p_2)}{4\mu L}\right)\left[1 - \left(\frac{r}{R_P}\right)^2\right]$

SOLUTION

The velocity components must satisfy the simplified continuity equation for a constant density fluid. This is Eq. 11.4a or 11.4b, depending on the coordinate system in use.

A. Since we have $u = u(y)$ and $(v = w = 0)$, substituting the three velocity components into Eq. 11.4a gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}(u(y)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 0$$

Equation 11.4a is satisfied, so this flow is possible. In fact, you may have recognized it as channel flow.

B. In this case the relevant equation is Eq. 11.4b:

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Since we are told $v_r = 0$ and $v_\theta = 0$, this reduces to $\partial v_z/\partial z = 0$, which can be seen to be satisfied by inspection. This flow is also possible; in fact, it is Poiseuille flow in a round pipe.

while in cylindrical coordinates the requirement is

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad (11.4b)$$

11.3 MOMENTUM EQUATION

CD/Dynamics/Newton's second law of motion/The momentum equation

The partial differential equation expressing conservation of momentum for a Newtonian fluid is called the Navier–Stokes equation. The derivation of this vector equation also

begins by applying the Reynolds transport theorem to a material volume of fluid in the form:

$$\frac{dE_{sys}}{dt} = \int_{R(t)} \frac{\partial}{\partial t}(\rho\varepsilon)\,dV + \int_{S(t)} (\rho\varepsilon)\mathbf{u}\cdot\mathbf{n}\,dS$$

In this case E_{sys} is chosen to be the total linear momentum in the volume, for which the intensive counterpart is the linear momentum per unit mass $\varepsilon = \mathbf{u}$. By Newton's law, the time rate of change of linear momentum within the material volume equals the sum of the body and surface forces acting on the volume:

$$\int_{R(t)} \frac{\partial}{\partial t}(\rho\mathbf{u})\,dV + \int_{S(t)} (\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}\,dS = \int_{R(t)} \rho\mathbf{f}\,dV + \int_{S(t)} \mathbf{\Sigma}\,dS$$

Note that the momentum equation for a material volume is identical to Eq. 7.18 for a CV; moreover, we have used the fact that the total body force is given as usual by the volume integral (Eq. 4.7) $\mathbf{F}_B = \int_{R(t)} \rho\mathbf{f}\,dV$, while the total surface force is given by the surface integral (Eq. 4.21) $\mathbf{F}_S = \int_{S(t)} \mathbf{\Sigma}\,dS$. To derive the differential momentum equation, we will write the surface integral in terms of the stress tensor rather than the stress vector, using Eq. 4.32 to write $\mathbf{F}_S = \int_S (\mathbf{n}\cdot\boldsymbol{\sigma})\,dS$. Next we use Gauss's theorem to write the surface integral in terms of the volume integral of the stress divergence $(\nabla\cdot\boldsymbol{\sigma})$, as defined by Eqs. 4.39a–4.39c in Section 4.7. Thus, the surface force is now given by the volume integral $\mathbf{F}_S = \int_{R(t)} (\nabla\cdot\boldsymbol{\sigma})\,dV$. Substituting this result into the momentum equation for a material volume, we have

$$\int_{R(t)} \frac{\partial}{\partial t}(\rho\mathbf{u})\,dV + \int_{S(t)} (\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}\,dS = \int_{R(t)} \rho\mathbf{f}\,dV + \int_{R(t)} (\nabla\cdot\boldsymbol{\sigma})\,dV$$

To derive a differential equation, we use Gauss's theorem to transform the flux integral into a volume integral, noting that we need the tensor form because $(\rho\mathbf{u})\mathbf{u}$ is a tensor. Next, we combine all the volume integrals into one, obtaining

$$\int_{R(t)} \left(\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \rho\mathbf{f} - \nabla\cdot\boldsymbol{\sigma}\right) dV = 0$$

Since the volume is arbitrary, the integrand must be zero. Thus the differential equation expressing the law of momentum conservation is given by

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) = \rho\mathbf{f} + \nabla\cdot\boldsymbol{\sigma}$$

The preceding equation is referred to as the conservative form of the differential momentum equation. This form serves as the starting point in many numerical algorithms used to solve the governing equations in computational fluid dynamics.

The traditional form of the momentum equation is obtained by expanding the time derivative and divergence terms, then rearranging the remaining terms to obtain

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \mathbf{u}\left[\frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u}\right] = \rho\mathbf{f} + \nabla\cdot\boldsymbol{\sigma}$$

The term in the square bracket is the left-hand side of the continuity equation (see Eq. 11.1b). Thus this term is equal to zero, and we can write the differential momentum equation as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = \rho \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}$$

Using the material derivative, the differential momentum equation takes its traditional form:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}$$

In deriving the momentum equation, we have employed the fluid density, fluid velocity, and stress tensor as variables but have not restricted the discussion to a certain type of fluid. Thus the momentum equation, like the continuity equation, is applicable to all fluids, compressible and incompressible, Newtonian and non-Newtonian, and for the whole range of flow speeds. As one of the three fundamental governing equations of fluid mechanics, it expresses the law of conservation of momentum at each point in the fluid. Although every physically possible fluid flow must satisfy this equation, it cannot be solved unless we introduce a constitutive model that provides relationships between the stress tensor and the velocity field. We will discuss the constitutive model for a Newtonian fluid in the next section.

Let us now consider what the various terms in the momentum equation represent. Recalling that the material derivative of velocity defines the fluid acceleration, we see that the left-hand side of the momentum equation is the product of density and fluid acceleration. Thus we could write the momentum equation as

$$\rho \mathbf{a} = \rho \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}$$

The left-hand side of this equation represents the inertial force per unit volume. The two terms on the right represent the body and surface forces per unit volume (the latter in terms of the stress divergence). Thus, the momentum equation represents a balance of inertial, body, and surface force per unit volume at each point in a fluid. You may find it worthwhile at this point to reread Section 4.7 and review the effects of stress variation in a fluid.

Using the definition of the stress divergence, Eqs. 4.39, we can write the three components of the momentum equation in Cartesian coordinates as

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_x + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right) \quad (11.5a)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho f_y + \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\right) \quad (11.5b)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho f_z + \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \quad (11.5c)$$

EXAMPLE 11.4

Consider an infinitesimal CV filled with fluid as shown in Figure 11.3A. Apply a momentum balance in the x direction to this CV to derive the x component of the momentum equation in Cartesian coordinates.

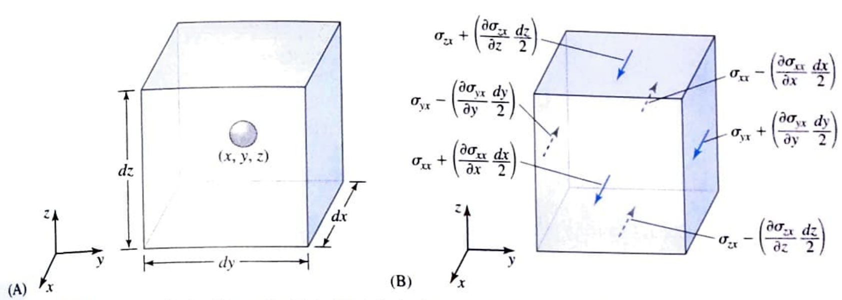

Figure 11.3 Schematic for Example 11.4: (A) infinitesimal fluid volume and (B) stress values.

SOLUTION

We are asked to derive the x component of the momentum equation in Cartesian coordinates for a specified volume of fluid. Figure 11.3 serves as the sketch for this system. Recalling the procedure used to perform a mass balance for this CV in Example 11.2, we will use a Taylor series expansion to relate the values of density, velocity, and stress on each face to the values at the center of this cube. To apply a momentum balance, we write Eq. 7.18):

$$\int_{CV} \frac{\partial}{\partial t}(\rho \mathbf{u})\, dV + \int_{CS} (\rho \mathbf{u})(\mathbf{u} \cdot \mathbf{n})\, dS = \int_{CV} \rho \mathbf{f}\, dV + \int_{CS} \boldsymbol{\Sigma}\, dS$$

As discussed earlier, we will write the surface force in terms of the stress tensor rather than the stress vector, using Eq. 4.32, $\mathbf{F}_S = \int_S (\mathbf{n} \cdot \boldsymbol{\sigma})\, dS$. The resulting momentum balance is

$$\int_{CV} \frac{\partial}{\partial t}(\rho \mathbf{u})\, dV + \int_{CS} (\rho \mathbf{u})(\mathbf{u} \cdot \mathbf{n})\, dS = \int_{CV} \rho \mathbf{f}\, dV + \int_{CS} (\mathbf{n} \cdot \boldsymbol{\sigma})\, dS$$

The x component of this equation is $\int_{CV}(\partial/\partial t)(\rho u)\, dV + \int_{CS}(\rho u)(\mathbf{u} \cdot \mathbf{n})\, dS = \int_{CV} \rho f_x\, dV + \int_{CS} (\mathbf{n} \cdot \boldsymbol{\sigma})_x dS$, where the integrand of the stress integral, $(\mathbf{n} \cdot \boldsymbol{\sigma})_x$, gives the stresses that act on the faces in the x direction as shown in Figure 11.3B. Notice in this figure that a first order Taylor series expansion has been used to relate the value of a stress on a face to the value at the center of the cube.

Each of the integrals in the x component momentum balance has a constant integrand. We can therefore write the volume integrals in terms of the values of the integrand at the center of the cube multiplied by the volume of the cube $(dx\,dy\,dz)$, to obtain

$$\int_{CV} \frac{\partial}{\partial t}(\rho u)\,dV = \frac{\partial}{\partial t}(\rho u)\,dx\,dy\,dz = \left(\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t}\right)dx\,dy\,dz \tag{A}$$

and

$$\int_{CV} \rho f_x\,dV = \rho f_x\,dx\,dy\,dz \tag{B}$$

The stress integral $\int_{CS}(\mathbf{n}\cdot\boldsymbol{\sigma})_x\,dS = (\mathbf{n}\cdot\boldsymbol{\sigma})_x A$ is evaluated by using the stress values shown in Figure 11.3B and considering each pair of faces in turn. For example, on the near and far faces we find, respectively, $(\mathbf{n}\cdot\boldsymbol{\sigma})_x A = [\sigma_{xx} + (\partial\sigma_{xx}/\partial x)(dx/2)]\,dy\,dz$ and $(\mathbf{n}\cdot\boldsymbol{\sigma})_x A = -[\sigma_{xx} - (\partial\sigma_{xx}/\partial x)(dx/2)]\,dy\,dz$. The net surface force on this pair of faces is therefore $(\mathbf{n}\cdot\boldsymbol{\sigma})_x A = (\partial\sigma_{xx}/\partial x)\,dx\,dy\,dz$. The contribution from the remaining two pairs of faces is found to be $(\partial\sigma_{yx}/\partial y)\,dx\,dy\,dz + (\partial\sigma_{zx}/\partial z)\,dx\,dy\,dz$, thus the total surface force on the cube is to first order

$$\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right)dx\,dy\,dz \tag{C}$$

The momentum flux integral is of the form $\int_{CS}(\rho u)(\mathbf{u}\cdot\mathbf{n})\,dS = (\rho u)(\mathbf{u}\cdot\mathbf{n})A$ and may be evaluated by using a Taylor series expansion (see Example 11.2) to define the appropriate values of ρ, u, $(\mathbf{u}\cdot\mathbf{n})$ and the area A on the six faces. For example, the momentum flux on the near face, where $\rho_{near} = \rho + (\partial\rho/\partial x)(dx/2)$ and $(\mathbf{u}\cdot\mathbf{n})$ is given by $+u_{near} = u + (\partial u/\partial x)(dx/2)$, takes the form

$$(\rho u)(\mathbf{u}\cdot\mathbf{n})A = \left[\rho + \left(\frac{\partial\rho}{\partial x}\right)\left(\frac{dx}{2}\right)\right]\left[u + \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{2}\right)\right]\left[u + \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{2}\right)\right]dy\,dz$$

$$= \rho uu\,dy\,dz + \left(\rho u\frac{\partial u}{\partial x} + \frac{1}{2}uu\frac{\partial\rho}{\partial x}\right)dx\,dy\,dz \quad \text{to first order}$$

where we have neglected higher order terms as usual. On the far face, where $(\mathbf{u}\cdot\mathbf{n})$ takes the value $-u_{far} = -u + (\partial u/\partial x)(dx/2)$, we find

$$(\rho u)(\mathbf{u}\cdot\mathbf{n})A = \left[\rho - \left(\frac{\partial\rho}{\partial x}\right)\left(\frac{dx}{2}\right)\right]\left[u - \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{2}\right)\right]\left[-u + \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{2}\right)\right]dy\,dz$$

$$= -\rho uu\,dy\,dz + \left(\rho u\frac{\partial u}{\partial x} + \frac{1}{2}uu\frac{\partial\rho}{\partial x}\right)dx\,dy\,dz \quad \text{to first order}$$

The sum of these two terms is

$$\left(2\rho u\frac{\partial u}{\partial x} + uu\frac{\partial\rho}{\partial x}\right)dx\,dy\,dz$$

or equivalently

$$\left[\rho u\frac{\partial u}{\partial x} + u\left(\rho\frac{\partial u}{\partial x} + u\frac{\partial\rho}{\partial x}\right)\right]dx\,dy\,dz$$

The two remaining pairs of faces contribute fluxes of

$$\left[\rho v\frac{\partial u}{\partial y} + u\left(\rho\frac{\partial v}{\partial y} + v\frac{\partial\rho}{\partial y}\right)\right]dx\,dy\,dz \quad \text{and} \quad \left[\rho w\frac{\partial u}{\partial z} + u\left(\rho\frac{\partial w}{\partial z} + w\frac{\partial\rho}{\partial z}\right)\right]dx\,dy\,dz$$

Thus, after some rearrangement, the total momentum flux is to first order

$$\left\{\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + u\left[\left(u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}\right) + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]\right\}dx\,dy\,dz \tag{D}$$

Gathering terms A–D, rearranging, and dividing by the common factor $dx\,dy\,dz$ yields

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + u\left[\left(\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}\right) + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$$

$$= \rho f_x + \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right)$$

The final step is to realize that since the term in square brackets is the continuity equation in the form of Eq. 11.2a, it has a value of zero. Therefore the final result is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_x + \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right)$$

which is identical to Eq. 11.5a, as expected.

11.4 CONSTITUTIVE MODEL FOR A NEWTONIAN FLUID

Examination of the three Cartesian components of the momentum equation (11.5a–11.5c) shows that they involve three velocity components, temporal and spatial derivatives of these velocity components, and spatial derivatives of the six independent components of the stress tensor. In their present form these governing equations are incomplete: there are too many unknowns and not enough equations. What is missing is a relationship between stress and rate of strain for the particular fluid involved. This relationship is part of what is known as a constitutive model for a fluid. The key function of a constitutive model is to provide the necessary relationships between the components

We conclude that the derivatives of the various stresses obey the following equations:

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right) = 0$$

$$\rho(-g) + \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\right) = 0$$

$$\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) = 0$$

Inserting the stresses found earlier into these reduced momentum equations, we find

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right) = \left[\frac{\partial}{\partial x}(-p) + \frac{\partial}{\partial y}\left(\frac{\mu U}{h}\right) + \frac{\partial}{\partial z}(0)\right] = -\frac{\partial p}{\partial x} = 0$$

$$\rho(-g) + \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\right) = \rho(-g) + \left[\frac{\partial}{\partial x}\left(\frac{\mu U}{h}\right) + \frac{\partial}{\partial y}(-p) + \frac{\partial}{\partial z}(0)\right]$$

$$= -\rho g - \frac{\partial p}{\partial y} = 0$$

$$\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) = \left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-p)\right] = -\frac{\partial p}{\partial z} = 0$$

From the first and last of these equations we conclude that the pressure does not vary in the x or z directions. Integrating the remaining equation $\partial p/\partial y = -\rho g$, noting that the density is constant, and evaluating the constant of integration on the top plate at $y = h$, we find $p(y) = p_h - \rho g(y - h)$. Thus, the momentum equations have shown that the pressure distribution in this shear flow is unchanged from the hydrostatic pressure distribution that would exist in the absence of flow. Notice that both the momentum equations and the constitutive relations are needed to solve this (typical) flow problem.

Did you recognize that the flow in this last example is the basis for the definition of viscosity? Notice that the fluid is sheared in the thin gap between parallel plates, and since v and w are zero, we have $\tau = \sigma_{yx} = \mu(\partial u/\partial y + \partial v/\partial x) = \mu(\partial u/\partial y)$, which is the defining equation used in our discussion of Newton's law of viscosity (Eq. 1.2c) in Chapter 1.

11.5 NAVIER–STOKES EQUATIONS

CD/Dynamics/Navier–Stokes equations

When the constitutive relationships for a Newtonian fluid (Eqs. 11.6) are used to replace the stresses in the differential momentum equations (Eqs. 11.5), the result is the Navier–Stokes equations in Cartesian coordinates. These equations, which describe the behavior of a Newtonian fluid with variable density and viscosity, are applicable to laminar and turbulent flows of liquids and gases throughout the entire range of

flow speeds. They are the basic equations that all commercial CFD codes employ to solve the most general class of fluid mechanics problems. The Navier–Stokes equations are exceedingly complicated and difficult to solve. Since the flows of interest in this text can be modeled as having constant density and constant viscosity, some simplification is, however, possible. In this case, the continuity equation is given by Eq. 11.3, $\nabla \cdot \mathbf{u} = 0$, and spatial derivatives of the (constant) viscosity are zero. Without going into all the details, we can write the continuity and Navier–Stokes equations for the important case of a constant density, constant viscosity fluid as (Eq. 11.4a) $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$, and

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (11.10a)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho f_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (11.10b)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho f_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad (11.10c)$$

In interpreting these equations, we note that they are actually the three components of a force balance on the fluid. We can write this balance in vector form as

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{f} - \nabla p + \mu\nabla^2\mathbf{u} \quad (11.11)$$

You should be able to recognize that the inertial forces per unit volume, given by $\rho\mathbf{a} = \rho(D\mathbf{u}/Dt)$, are balanced by the sum of body forces per unit volume, $\rho\mathbf{f}$, the pressure forces per unit volume as given by $-\nabla p$, and viscous forces per unit volume as given by $\mu\nabla^2\mathbf{u}$. Thus, the vector equation $\rho(D\mathbf{u}/Dt) = \rho\mathbf{f} - \nabla p + \mu\nabla^2\mathbf{u}$ is another way to write the Navier–Stokes equation for a constant density, constant viscosity fluid.

CD/Dynamics/Newton's second law of motion/$F = ma$ for a Newtonian Fluid

In cylindrical coordinates, the continuity and Navier–Stokes equations for a constant density, constant viscosity fluid are $(1/r)\partial(rv_r)/\partial r + (1/r)\partial v_\theta/\partial\theta + \partial v_z/\partial z = 0$, which is Eq. (11.4b), and

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right)$$

$$= \rho f_r - \frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) \quad (11.12a)$$

HISTORY BOX 11.1

Claude Navier (1785–1836) entered the prestigious École Polytechnique in 1802 as a marginal student but emerged at the top of his class. He inherited the role of the leading scholar of mathematics, science, and engineering in France from his teacher and friend Jean Baptiste Fourier. In 1822 he presented a paper that for the first time accurately described the role of friction in the equations of motion for a fluid. His analysis started from a molecular view of fluid. It was left to Jean-Claude de Saint-Venant to explain this result based on the viscous stresses in the fluid and to identify the viscosity as the key material property.

CD/History/Claude Navier and Sir George Stokes

George Stokes (1819–1903) held the Lucasian chair at Cambridge University, the same position held by Sir Isaac Newton. Thus, he was one of the leading scholars in England. He made many contributions to fluid mechanics and the nature of light. With no knowledge of the work in France, in 1845 Stokes published a derivation of the equations that bear his name, using an analysis based on the internal friction of fluid much like we have presented here.

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right)$$
$$=\rho f_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) \tag{11.12b}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$
$$=\rho f_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right) \tag{11.12c}$$

The continuity and Navier–Stokes equations just given provide a complete set of governing equations to determine the velocity and pressure at every point in a flow. It is not necessary in this case to solve the energy equation to determine the velocity and pressure fields, because we have four equations and four unknowns: three components of velocity and the pressure.

CFD

We have often mentioned that solving the governing equations is a difficult task. In Chapter 12 we will demonstrate how to construct analytical solutions for a number of important flows. Here we emphasize that saying we have obtained a solution to the governing equations means one of two things. In an analytical solution, the functions describing the velocity and pressure fields must satisfy all four equations simultaneously, as well as the boundary conditions, when the functions are inserted into the equations

and the various spatial derivatives evaluated. The next two examples demonstrate this process. In a CFD solution, the set of numerical values for velocity and pressure constituting the solution on some number of spatial points satisfies a massive set of algebraic equations representing the discretized form of the governing equations and boundary conditions to some degree of approximation. This is about all we can say in general about a CFD solution because the details depend to a great degree on the type of approach used by the CFD model.

EXAMPLE 11.6

In the channel flow of a constant density, constant viscosity fluid shown in Figure 11.5, suppose the complete description of the flow is given in Cartesian coordinates by the velocity field $u=\{[h^2(p_1-p_2)]/2\mu L\}[1-(y/h)^2]$, $v=0$, and $w=0$, and the pressure field $p(x)=p_1+[(p_2-p_1)/L](x-x_1)$. Here p_1 and p_2 are the pressures at the indicated locations, and the slight hydrostatic variation in pressure across the channel has been ignored. Show that this flow satisfies the constant density, constant viscosity Navier–Stokes equations in Cartesian coordinates, with the body force neglected.

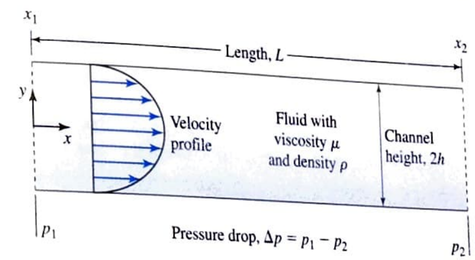

Figure 11.5 Schematic for Example 11.6.

SOLUTION

We will first substitute the three velocity components into the continuity equation for an incompressible fluid, Eq. 11.4a, then substitute the velocity components and pressure into the constant density, constant viscosity forms of the Navier–Stokes equations, Eqs. 11.10a–11.10c. Since we have $u=u(y)$ only, with v and w zero, the continuity equation, $\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0$, is satisfied by inspection. Terms in Eqs. 11.10a–11.10c that contain the velocity components v and w are zero, as are time derivatives and spatial derivatives of u with respect to x and z. Writing only the remaining nonzero terms, and setting the body force terms to zero, we find

$$0=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2},\quad 0=-\frac{\partial p}{\partial y},\quad \text{and}\quad 0=-\frac{\partial p}{\partial z}$$

The last two equations are consistent with the given pressure distribution. Substituting the x velocity component and pressure into the first pressure equation yields

$$0 = -\frac{\partial}{\partial x}\left(p_1 + \left(\frac{p_2 - p_1}{L}\right)(x - x_1)\right) + \mu\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\left(\frac{h^2(p_1 - p_2)}{2\mu L}\right)\left[1 - \left(\frac{y}{h}\right)^2\right]\right)\right]$$

$$0 = -\left(\frac{p_2 - p_1}{L}\right) + \mu\left(\frac{h^2(p_1 - p_2)}{2\mu L}\right)\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left[1 - \left(\frac{y}{h}\right)^2\right]\right)$$

$$0 = -\left(\frac{p_2 - p_1}{L}\right) + \mu\left(\frac{h^2(p_1 - p_2)}{2\mu L}\right)\left(\frac{-2}{h^2}\right) = 0$$

We see that the velocity and pressure do satisfy the appropriate forms of the continuity and Navier–Stokes equations. It is also straightforward to show that the velocity field satisfies the no-slip, no-penetration conditions at the channel walls.

EXAMPLE 11.7

In the Poiseuille flow of a constant density, constant viscosity fluid in a round pipe, (Figure 11.6), the velocity field is given in cylindrical coordinates by $\mathbf{u} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$ with components $v_r = 0$, $v_\theta = 0$, and $v_z(r) = \{[R_P^2(p_1 - p_2)]/4\mu L\}[1 - (r/R_P)^2]$. Find the pressure distribution in this flow if body forces are neglected.

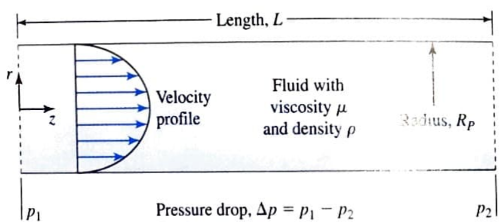

Figure 11.6 Schematic for Example 11.7.

SOLUTION

The velocity field must satisfy the continuity equation, Eq. 11.4b, and the velocity field and pressure distribution must satisfy Eqs. 11.12a–11.12c. We begin by checking the continuity equation:

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Since we know $v_r = 0$ and $v_\theta = 0$, this reduces to $\partial v_z/\partial z = 0$, which can be seen to be satisfied by inspection, since $v_z = v_z(r)$. Writing only the nonzero terms in Eqs. 11.12a–11.12c, we have

$$0 = -\frac{\partial p}{\partial r}, \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \quad \text{and} \quad 0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial^2 r} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right)$$

Thus the pressure is a function of z only. Inserting the given velocity component into the last equation and taking derivatives we find

$$\frac{\partial p}{\partial z} = \mu\left(\frac{R_P^2(p_1 - p_2)}{4\mu L}\right)\left[\left(-\frac{2}{R_P^2}\right) + \frac{1}{r}\left(-\frac{2r}{R_P^2}\right)\right]$$

$$= \mu\left(\frac{R_P^2(p_1 - p_2)}{4\mu L}\right)\left(-\frac{2}{R_P^2} - \frac{2}{R_P^2}\right)$$

which, after simplification, yields $\partial p/\partial z = (p_2 - p_1)/L$. Integrating and evaluating the resulting constant of integration at $z = z_1$, we find $p(z) = p_1 + [(p_2 - p_1)/L](z - z_1)$. This is a linear drop in pressure down the pipe. Can you see by inspection that the no-slip, no-penetration conditions are satisfied on the pipe wall?

CD/Dynamics/Potential Flow

11.6 EULER EQUATIONS

The formidable Navier–Stokes equations have generated many efforts to introduce simplifying approximations. One of the earliest and most valuable of these approximations is that of an inviscid fluid, defined in Chapter 8 to be a fluid whose viscosity is zero. Upon substituting $\mu = 0$ into the constitutive model for a Newtonian fluid (Eqs. 11.6a–11.6f), we find the state of stress in an inviscid fluid in Cartesian coordinates to be

$$\sigma_{xx} = -p, \quad \sigma_{yy} = -p, \quad \sigma_{zz} = -p \qquad (11.13a–c)$$

$$\sigma_{xy} = \sigma_{yx} = 0, \quad \sigma_{zy} = \sigma_{yz} = 0, \quad \sigma_{zx} = \sigma_{xz} = 0 \qquad (11.13d–f)$$

As expected, we see that an inviscid fluid is incapable of exerting a shear stress. This is another way to define an inviscid fluid. The absence of shear stress indicates that an inviscid fluid does not obey the no-slip condition and therefore must slip along a solid surface. The state of stress in an inviscid fluid, with its absence of shear stresses, is given by a pressure distribution alone, just as is the case in a static fluid. However, as we will see in a moment, in an inviscid fluid, the pressure distribution is related to both the body force and the inertial force created by the velocity field, rather than to just the body force as in a static fluid.

For problems that involve the mixing of different fluids, or the transport of some substance in a fluid, the Reynolds transport theorem may be used to derive additional governing equations that describe the mixing or transport processes involved. This task is beyond the scope of this introductory text, but the process of deriving these equations is very similar to that described in this chapter for mass, momentum, and energy.

Since it is tedious, we will not write out these two forms of the energy equation in Cartesian coordinates, but you will find these equations written in various coordinate systems in many advanced texts.

The continuity, Navier–Stokes, and energy equations, together with the Newtonian constitutive model, constitute a complete mathematical description of fluid flow. CFD codes use these three equations as the starting point for describing fluid flows. As mentioned earlier, however, it is not necessary to solve the energy equation in the flow of a constant density, constant viscosity fluid. In that case, the continuity and Navier–Stokes equations are sufficient to determine the velocity and pressure fields. However, if a flow of this type involves heat transfer, one may find the temperature distribution in the fluid by solving the energy equation after the velocity and pressure fields have been obtained.

11.8 DISCUSSION

We conclude this chapter with a brief discussion of several concepts that have been mentioned in earlier chapters and have direct relevance to solving the governing equations.

CD/Dynamics/Boundary conditions

11.8.1 Initial and Boundary Conditions

The continuity, Navier–Stokes, and energy equations, together with the appropriate constitutive relationships and state equations, provide a complete mathematical description of the flow of a Newtonian fluid. To obtain a solution of this complex set of governing equations, we must specify an appropriate set of boundary and initial conditions for the flow problem being analyzed. The basic set of unknowns for which a solution is sought in the general case includes the three components of velocity, pressure, density, and the temperature of the fluid.

A complete discussion of the required boundary conditions depends on the exact nature of the problem, the approximations employed, and the set of equations to be solved. Although this discussion is beyond the scope of this text, the boundary conditions associated with the fluid velocity field are generally the no-slip, no-penetration conditions as discussed in Section 6.6. In an unsteady flow problem, the initial conditions take the form of the specification of the spatial distribution of the unknowns (velocity, pressure, etc.) at an initial instant of time. The selection of appropriate boundary and initial conditions will be demonstrated in Chapter 12, where a number of analytical solutions to simplified forms of the governing equations will be discussed.

CD/Dynamics/Reynolds number: Inertia and viscosity/Scaling the Navier–Stokes Equation

11.8.2 Nondimensionalization

In Section 3.2 we described many of the common dimensionless groups in fluid mechanics including the Reynolds number, $Re = \rho V L/\mu$, the Froude number $Fr = V/\sqrt{gL}$, the Euler number $Eu = (p - p_0)/\frac{1}{2}\rho V^2$, and the Strouhal number $St = \omega L/V$. We showed that these and other dimensionless groups naturally occur in applying dimensional analysis (DA) to flow problems. In this section we extend our discussion of DA to describe the process known as nondimensionalization of the governing equations. The value of this process is that complete similitude between two physical systems (e.g., a prototype and the full-scale device of interest) is guaranteed if the dimensionless governing equations and boundary conditions for the two different systems are identical. Another advantage of the dimensionless form of the governing equations is that a solution is applicable over a range of geometric and flow parameters, provided the values of those parameters leave the dimensionless coefficients in the governing equations unchanged.

Nondimensionalization of a governing equation is accomplished by dividing every dependent and independent variable in the equation by an appropriate combination of characteristic dimensions, thereby making each variable dimensionless. A characteristic dimension is a physical dimension that is in some way characteristic of the flow field under investigation. Common examples of characteristic dimensions include a characteristic length scale L, usually derived from the geometry; a characteristic velocity scale U, usually defined as the average fluid velocity; a characteristic pressure P; and a characteristic time scale T.

We will illustrate the process of obtaining nondimensional governing equations for the case of a constant density, constant viscosity, flow of a Newtonian fluid. If we assume that gravity is the only body force, and take the z axis upward as usual, the continuity and Navier–Stokes equations, (Eqs. 11.4a and 11.10a–c), are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\rho g - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Recall that for this case it is not necessary to solve the energy equation to determine the velocity and pressure fields.

INTRODUCTION TO

FLUID MECHANICS

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