Lecture 11 Fast Fourier Transform (FFT)

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Outline

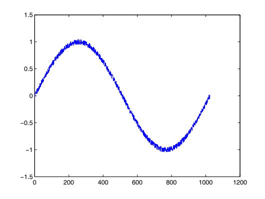
Examples

Fast Fourier Transform

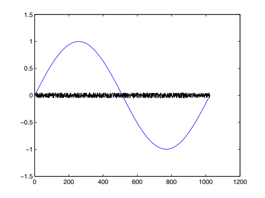
Applications

Signal processing

- Filtering: a polluted signal



- High pass and low pass filter (signal and noise)



- How to obtain the high frequency and low frequency quickly?

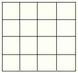
Solving PDEs on rectangular mesh

- Solving the Poisson equations

$$-\Delta u = f \quad \text{in} \quad \Omega$$

$$u = 0 \quad \text{on} \quad \partial\Omega$$

in the rectangular domain



- After discretization we will obtain the linear system with about N^2 unknowns

$$-\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{4h^2} = f_{ij}$$

- The FFT would give a fast algorithm to solve the system above with computational efforts $O(N^2 \log_2 N)$.

Computing convolution (卷积)

- Suppose
$$h(x) = \int_0^{2\pi} f(x-y)g(y)dy$$
is the convolution of f and g, where $f(x), g(x) \in C_{2\pi}$ are period 2π functions.
- Take $x_j = j\delta$, $j = 0, 1, \ldots, N-1$, $\delta = \frac{2\pi}{N}$ and apply simple rectangular discretization
$$h(x_i) \approx \sum_{j=0}^{N-1} f(x_i - x_j)g(x_j) \cdot \delta \qquad i = 0, 1, \ldots, N-1$$
- Define $f_i = f(x_i), g_i = g(x_i)$, and let f_i is period N respect to the subscript i, define
$$h_i = \sum_{j=0}^{N-1} f_{i-j} g_j \cdot \delta \qquad i = 0, 1, \ldots, N-1$$
- The direct computation is $O(N^2)$.

Fast Fourier Transform

Fast Fourier Transform is one of the top 10 algorithms in 20th century.

But its idea is quite simple, even for a high school student!

Outline

Fourier Transform

- Suppose $f(x)$ is absolutely integrable in $(-\infty, +\infty)$, then the Fourier transform of $f(x)$ is

$$\hat{f}(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx.$$

- Moreover if $f(x)$ is square integrable, then the inverse Fourier transform of $\hat{f}(k)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k)e^{ikx}dk.$$

Properties of Fourier transform

Some important properties of Fourier transform:

1. Derivative to coefficient:

$$\widehat{(f'(x))}(k) = ik\hat{f}(k);$$

2. Translation property:

$$(f\widehat{(x-a)})(k) = e^{-ika}\hat{f}(k);$$

3. Convolution to multiplication:

$$\widehat{(f*g)}(k) = \hat{f}(k)\hat{g}(k);$$

   where $(f*g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy$.

4. Parseval's identity:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{f}(k)|^2 dk.$$

Discrete Fourier transform (DFT)

- Suppose we have $\boldsymbol{a} = (a_0, a_1, \cdots, a_{N-1})^T$, define DFT of $\boldsymbol{a}$ as $\boldsymbol{c} = (c_0, c_1, \cdots, c_{N-1})^T \triangleq \hat{\boldsymbol{a}}$, where

$$c_k = \sum_{j=0}^{N-1} a_j e^{-jk\frac{2\pi i}{N}}, \qquad k = 0, 1, \ldots, N-1.$$

  Here i is the imaginary unit, $e^{-\frac{2\pi i}{N}} \triangleq \omega$ is the N-th root of unity.

- $\boldsymbol{a}$ is the inverse discrete Fourier transform of $\boldsymbol{c}$ defined as

$$a_j = \frac{1}{N}\sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi i}{N}}, \qquad j = 0, 1, \ldots, N-1.$$

- DFT is closely related to the trigonometric interpolation for 2π-periodic function

$$T(x) = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} b_k e^{ikx}.$$

  such that at $x_j = \frac{2j\pi}{N}$, $T(x_j) = a_j, \quad j = 0, 1, \ldots, N-1.$ The readers may find the relation between c_k and b_k.

Remark on DFT

- DFT can be considered as a linear transformation.
- Define Fourier matrix

$$F = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{pmatrix} = (\omega^{jk})_{j,k=0}^{N-1}$$

where ω is the N-th root of unity.

- $\boldsymbol{c}$ is the Fourier transform of $\boldsymbol{a}$ can be represented as

$$\mathbf{c} = F\mathbf{a}$$

Remark on DFT

- Inverse DFT can also be considered as a linear transformation.
- Define inverse Fourier matrix

$$F^{-1} = G = \frac{1}{N}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \cdots & \omega^{-(N-1)} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \omega^{-(N-1)} & \cdots & \omega^{-(N-1)^2} \end{pmatrix} = (\omega^{-jk})_{j,k=0}^{N-1}$$

where ω is the N-th root of unity.

- $\boldsymbol{a}$ is the inverse Fourier transform of $\boldsymbol{c}$ can be represented as

$$\boldsymbol{a} = G\boldsymbol{c}$$

Properties of DFT

- Convolution to multiplication:

$$\widehat{(f*g)}_k = \hat{f}_k \hat{g}_k \quad k = 0, 1, \ldots, N-1$$

where

$$(f*g)_l = \sum_{j=0}^{N-1} f_{l-j} g_j \quad l = 0, 1, \ldots, N-1,$$

and f_l is period N with respect to index l, i.e.

$$f_{-1} = f_{N-1}, f_{-2} = f_{N-2}, \ldots$$

- Parseval's identity:

$$N \sum_{j=0}^{N-1} |a_j|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

FFT idea

- FFT is proposed by J.W. Cooley and J.W. Tukey in 1960s, but the idea may be traced back to Gauss.
- The basic motivation is if we compute DFT directly, i.e.

$$\boldsymbol{c} = F\boldsymbol{a}$$

we need N^2 multiplications and $N(N-1)$ additions. Is it possible to reduce the computation effort?

- First consider the case $N = 4$

$$F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}, \quad F\mathbf{a} = \begin{pmatrix} (a_0 + a_2) + (a_1 + a_3) \\ (a_0 - a_2) - i(a_1 - a_3) \\ (a_0 + a_2) - (a_1 + a_3) \\ (a_0 - a_2) + i(a_1 - a_3) \end{pmatrix}$$

FFT idea

- From the concrete form of DFT, we actually need 2 multiplications (timing $\pm i$) and 8 additions ($a_0 + a_2$, $a_1 + a_3$, $a_0 - a_2$, $a_1 - a_3$ and the additions in the middle).
- This observation may reduce the computational effort from $O(N^2)$ into

$$O(N \log_2 N)$$

- Because

$$\lim_{N\to\infty} \frac{\log_2 N}{N} = 0$$

  It is a typical fast algorithm.
- Fast algorithms of this type of recursive halving are very typical in scientific computing.

Construction of FFT

- Consider $N = 2^m$ and denote

$$p(x) = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1},$$

divide $p(x)$ into odd (奇) and even (偶) power parts

$$\begin{aligned} p(x) &= (a_0 + a_2 x^2 + \cdots) + x(a_1 + a_3 x^2 + \cdots) \\ &= p_e(x^2) + x p_o(x^2) \end{aligned}$$

where

$$p_e(t) = a_0 + a_2 t + \ldots + a_{N-2} t^{\frac{N}{2}-1}, p_o(t) = a_1 + a_3 t + \ldots + a_{N-1} t^{\frac{N}{2}-1}$$

- Define $\omega_k = e^{-\frac{2\pi i}{k}}$ (k-th root of unity), then when $j = 0, 1, \ldots, \frac{N}{2} - 1$

$$\begin{cases} c_j & = p_e(\omega_N^{2j}) + \omega_N^j p_o(\omega_N^{2j}) \\ c_{\frac{N}{2}+j} & = p_e(\omega_N^{2(\frac{N}{2}+j)}) + \omega_N^{\frac{N}{2}+j} p_o(\omega_N^{2(\frac{N}{2}+j)}) \end{cases}$$

Construction of FFT

- Pay attention that

$$\omega_N^{2j} = \omega_{\frac{N}{2}}^{j}, \quad \omega_N^{\frac{N}{2}+j} = -\omega_N^{j}, \quad \omega_N^{N+2j} = \omega_{\frac{N}{2}}^{j}$$

 then

$$c_j = v_j + \omega_N^j u_j, c_{j+\frac{N}{2}} = v_j - \omega_N^j u_j \qquad j = 0, 1, ..., \frac{N}{2} - 1$$

 where

$$v_j = p_e(\omega_{\frac{N}{2}}^{j}), u_j = p_o(\omega_{\frac{N}{2}}^{j})$$

- The formula above show that the DFT of N components vector $\boldsymbol{a}$ could be converted to compute the DFT of two $\frac{N}{2}$ components vectors $\boldsymbol{a}_e, \boldsymbol{a}_o$ and some simple additions and multiplications. This is called Danielson-Lanczos algorithm. The recursive application of this idea gives FFT.

A simple example: $N = 8$

- Suppose the array

$$\boldsymbol{a} = (a_0, a_1, \cdots, a_7)^T$$

Step A: Splitting (reordering) (odd parts and even parts):

- Step 1

$$\boldsymbol{a}_e = (a_0, a_2, a_4, a_6)^T, \qquad \boldsymbol{a}_o = (a_1, a_3, a_5, a_7)^T;$$

- Step 2

$$\boldsymbol{a}_{ee} = (a_0, a_4)^T, \qquad \boldsymbol{a}_{eo} = (a_2, a_6)^T,$$
$$\boldsymbol{a}_{oe} = (a_1, a_5)^T, \qquad \boldsymbol{a}_{oo} = (a_3, a_7)^T;$$

- Step 3

$\boldsymbol{a}_{eee}$	$\boldsymbol{a}_{eeo}$	$\boldsymbol{a}_{eoe}$	$\boldsymbol{a}_{eoo}$	$\boldsymbol{a}_{oee}$	$\boldsymbol{a}_{oeo}$	$\boldsymbol{a}_{ooe}$	$\boldsymbol{a}_{ooo}$
a_0	a_4	a_2	a_6	a_1	a_5	a_3	a_7

A simple example: $N = 8$

Step B: Combination:

- Step 1

$$\boldsymbol{c}_{ee} = (a_0 + \omega_2^0 a_4,\ a_0 - \omega_2^0 a_4)^T,$$
$$\boldsymbol{c}_{eo} = (a_2 + \omega_2^0 a_6,\ a_2 - \omega_2^0 a_6)^T,$$
$$\boldsymbol{c}_{oe} = (a_1 + \omega_2^0 a_5,\ a_1 - \omega_2^0 a_5)^T,$$
$$\boldsymbol{c}_{oo} = (a_3 + \omega_2^0 a_7,\ a_3 - \omega_2^0 a_7)^T,$$

- Define the notations

$$\boldsymbol{w}_4 \triangleq (w_4^0, w_4^1)^T, \boldsymbol{w}_8 \triangleq (w_8^0, w_8^1, w_8^2, w_8^3)^T,$$

and

$$X \circ Y \triangleq (x_j y_j)_j$$

as the vector product through multiplication by components.

A simple example: $N = 8$

Step B: Combination:

- Step 2

$$\boldsymbol{c}_e = \begin{bmatrix} \boldsymbol{c}_{ee} + \boldsymbol{w}_4 \circ \boldsymbol{c}_{eo} \\ \boldsymbol{c}_{ee} - \boldsymbol{w}_4 \circ \boldsymbol{c}_{eo} \end{bmatrix}, \qquad \boldsymbol{c}_o = \begin{bmatrix} \boldsymbol{c}_{oe} + \boldsymbol{w}_4 \circ \boldsymbol{c}_{oo} \\ \boldsymbol{c}_{oe} - \boldsymbol{w}_4 \circ \boldsymbol{c}_{oo} \end{bmatrix},$$

- Step 3

$$\boldsymbol{c} = \begin{bmatrix} \boldsymbol{c}_e + \boldsymbol{w}_8 \circ \boldsymbol{c}_0 \\ \boldsymbol{c}_e - \boldsymbol{w}_8 \circ \boldsymbol{c}_0 \end{bmatrix}$$

A simple sketch of FFT ($N = 8$)

$$\boldsymbol{a} \xrightarrow{\text{reordering}} \begin{bmatrix} a_0 \\ a_4 \\ a_2 \\ a_6 \\ a_1 \\ a_5 \\ a_3 \\ a_7 \end{bmatrix} \xrightarrow{\text{combination 1}} \begin{bmatrix} \boldsymbol{c}_{ee} \\ \boldsymbol{c}_{eo} \\ \boldsymbol{c}_{eo} \\ \boldsymbol{c}_{oo} \end{bmatrix} \xrightarrow{\text{combination 2}} \begin{bmatrix} \boldsymbol{c}_{e} \\ \boldsymbol{c}_{o} \end{bmatrix} \xrightarrow{\text{combination 3}} \boldsymbol{c}$$

A remark on the reordering

If we map e to 0, and o to 1, we can find the binary representation of the indices after reordering is just the bit reversal before reordering

$$
\begin{array}{ccc}
0 = 000_2 & & 000_2 = 0 \\
1 = 001_2 & & 100_2 = 4 \\
2 = 010_2 & & 010_2 = 2 \\
3 = 011_2 & & 110_2 = 6 \\
4 = 100_2 & \xrightarrow{\text{Bit reversal}} & 001_2 = 1 \\
5 = 101_2 & & 101_2 = 5 \\
6 = 110_2 & & 011_2 = 3 \\
7 = 111_2 & & 111_2 = 7
\end{array}
$$

Outline

Examples

Fast Fourier Transform

Applications

Compute the convolution

- From the discretization at the beginning, we have

$$h_i = \sum_{j=0}^{N-1} f_{i-j} g_j \cdot \delta \qquad i = 0, 1, \ldots, N-1$$

  thus

$$\boldsymbol{h} = (\hat{\boldsymbol{h}})^\vee = (\delta \cdot \hat{\boldsymbol{f}} \circ \hat{\boldsymbol{g}})^\vee$$

- After using FFT, $N^2 + N$ multiplications and $N(N-1)$ additions are reduced to $\frac{3}{2} N \log_2 N + 2N$ multiplications and $3N \log_2 N$ additions.

Solving the linear system with loop matrix

- Let
$$L = \begin{pmatrix} c_0 & c_{N-1} & \cdots & c_1 \\ c_1 & c_0 & \cdots & c_2 \\ \cdots & \cdots & \cdots & \cdots \\ c_{N-1} & c_{N-2} & \cdots & c_0 \end{pmatrix}$$
Solving $L\boldsymbol{x} = \boldsymbol{b}$. L is a loop matrix.

- We have
$$(L\boldsymbol{x})_i = \sum_{j=0}^{N-1} c_{i-j} x_j$$
where we assume $\boldsymbol{c}$ is period N with respect to the subscripts, and $\boldsymbol{x} = (x_0, x_1, \ldots, x_{N-1})^T$.

Solving the linear system with loop matrix

- First consider the Jordan form of L. From the formula before

$$L\boldsymbol{x} = \boldsymbol{c} * \boldsymbol{x} = \lambda \boldsymbol{x}$$

Take DFT we have

$$\hat{\boldsymbol{c}} \circ \hat{\boldsymbol{x}} = \lambda \hat{\boldsymbol{x}}$$

then eigenvalues

$$\lambda_k = \hat{\boldsymbol{c}}_k$$

- The eigenvectors

$$\hat{\boldsymbol{x}}_j^{(k)} = \delta_{kj}, \quad (j, k = 0, 1, \ldots, N-1)$$

where δ_{kj} is Kronecker's δ.

- Take inverse transform we obtain

$$\begin{gathered} \boldsymbol{x}^{(0)} = (1, 1, \ldots, 1)^T, \\ \boldsymbol{x}^{(1)} = (1, \omega^{-1}, \ldots, \omega^{-(N-1)})^T, \\ \cdots\cdots\cdots \\ \boldsymbol{x}^{(N-1)} = (1, \omega^{-(N-1)}, \ldots, \omega^{-(N-1)^2})^T \end{gathered}$$

Solving the linear system with loop matrix

- Spectral decomposition of L

$$
\begin{aligned}
L &= (\boldsymbol{x}^{(0)}\boldsymbol{x}^{(1)}\cdots\boldsymbol{x}^{(N-1)})\begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{N-1} \end{pmatrix}(\boldsymbol{x}^{(0)}\boldsymbol{x}^{(1)}\cdots\boldsymbol{x}^{(N-1)})^{-1} \\
&= (NF^{-1})\Lambda\,(NF^{-1})^{-1} = F^{-1}\Lambda F
\end{aligned}
$$

- Solving $L\boldsymbol{x} = \boldsymbol{b}$ is equivalent to $F^{-1}\Lambda F\boldsymbol{x} = \boldsymbol{b}$, i.e. $\Lambda(F\boldsymbol{x}) = F\boldsymbol{b}$. Then it is composed of three steps:
  - Step 1: Compute $F\boldsymbol{b}$ i.e. apply FFT to $\boldsymbol{b}$ to obtain $\hat{\boldsymbol{b}}$;
  - Step 2: Compute Λ i.e. apply FFT to $\boldsymbol{c}$ to obtain $\hat{\boldsymbol{c}}$;
  - Step 3: Compute $\hat{x}_k = \hat{b}_k/\hat{c}_k$, and then compute $(\hat{\boldsymbol{x}})^{\vee}$ to obtain$\boldsymbol{x}$.

Homework assignment

- Familiarize the “FFT” and “IFFT” command in MATLAB;
- Compute the convolution for

$$h(x) = \int_0^{2\pi} \sin(x-y)e^{\cos y}dy$$